CS 766/QIC 820 Theory of Quantum Information (Fall 2011)

Lecture 19: LOCC and separable measurements

In this lecture we will discuss measurements that can be collectively performed by two parties by means of local quantum operations and classical communication. Much of this discussion could be generalized to measurements implemented by more than two parties—but, as we have been doing for the last several lectures, we will restrict our attention to the bipartite case.

19.1 Definitions and simple observations

Informally speaking, an LOCC measurement is one that can be implemented by two (or more) parties using only local quantum operations and classical communication. We must, however, choose a more precise mathematical definition if we are to prove mathematical statements concerning these objects.

There are many ways one could formally define LOCC measurements; for simplicity we will choose a definition that makes use of the definition of LOCC channels we have already studied. Specifically, we will say that a measurement

$$\mu : \Gamma \to \mathrm{Pos}\left(\mathcal{X}_A \otimes \mathcal{X}_B\right)$$

on a bipartite system having associated complex Euclidean spaces $\mathcal{X}_A$ and $\mathcal{X}_B$ is an *LOCC measurement* if there exists an LOCC channel

$$\Phi \in \mathrm{LOCC}\left(\mathcal{X}_A, \mathbb{C}^\Gamma : \mathcal{X}_B, \mathbb{C}\right)$$

such that

$$\langle E_{a,a}, \Phi(\rho)\rangle = \langle \mu(a), \rho\rangle \tag{19.1}$$

for every $a \in \Gamma$ and $\rho \in \mathrm{D}\left(\mathcal{X}_A \otimes \mathcal{X}_B\right)$. An equivalent condition to (19.1) holding for all $\rho \in \mathrm{D}\left(\mathcal{X}_A \otimes \mathcal{X}_B\right)$ is that

$$\mu(a) = \Phi^*(E_{a,a}).$$

The interpretation of this definition is as follows. Alice and Bob implement the measurement μ by first performing the LOCC channel Φ, which leaves Alice with a register whose classical states coincide with the set Γ of possible measurement outcomes, while Bob is left with nothing (meaning a trivial register, having a single state, whose corresponding complex Euclidean space is $\mathbb{C}$). Alice then measures her register with respect to the standard basis of $\mathbb{C}^\Gamma$ to obtain the measurement outcome. Of course there is nothing special about letting Alice perform the measurement rather than Bob; we are just making an arbitrary choice for the sake of arriving at a definition, which would be equivalent to one allowing Bob to make the final measurement rather than Alice.

As we did when discussing channels, we will also consider a relaxation of LOCC measurements that is often much easier to work with. A measurement

$$\mu : \Gamma \to \mathrm{Pos}\left(\mathcal{X}_A \otimes \mathcal{X}_B\right)$$

is said to be *separable* if, in addition to satisfying the usual requirements of being a measurement, it holds that $\mu(a) \in \text{Sep}\,(\mathcal{X}_A : \mathcal{X}_B)$ for each $a \in \Gamma$.

Proposition 19.1. *Let $\mathcal{X}_A$ and $\mathcal{X}_B$ be complex Euclidean spaces and let*

$$\mu : \Gamma \to \text{Pos}\,(\mathcal{X}_A \otimes \mathcal{X}_B)$$

be an LOCC measurement. It holds that μ is a separable measurement.

Proof. Let $\Phi \in \text{LOCC}\,(\mathcal{X}_A, \mathcal{Y}_A : \mathcal{X}_B, \mathcal{Y}_B)$ be an LOCC channel for which

$$\mu(a) = \Phi^*(E_{a,a})$$

for each $a \in \Gamma$. As Φ is an LOCC channel, it is necessarily separable, and therefore so too is Φ^*. (This may be verified by considering the fact that Kraus operators for Φ^* may be obtained by taking adjoints of the Kraus operators of Φ.) As $E_{a,a} = E_{a,a} \otimes 1$ is an element of $\text{Sep}\,(\mathbb{C}^\Gamma : \mathbb{C})$ for every $a \in \Gamma$, we have that $\mu(a) = \Phi^*(E_{a,a})$ is separable for each $a \in \Gamma$ as required. □

It is the case that there are separable measurements that are not LOCC measurements—we will see such an example (albeit without a proof) later in the lecture. However, separable measurements can be simulated by LOCC measurement in a probabilistic sense: the LOCC measurement that simulates the separable measurement might fail, but it succeeds with nonzero probability, and if it succeeds it generates the same output statistics as the original separable measurement. This is implied by the following theorem.

Theorem 19.2. *Suppose $\mu : \Gamma \to \text{Pos}\,(\mathcal{X}_A \otimes \mathcal{X}_B)$ is a separable measurement. There exists an LOCC measurement*

$$\nu : \Gamma \cup \{\text{fail}\} \to \text{Pos}\,(\mathcal{X}_A \otimes \mathcal{X}_B)$$

with the property that $\nu(a) = \gamma\mu(a)$, for each $a \in \Gamma$, for some real number $\gamma > 0$.

Proof. A general separable measurement $\mu : \Gamma \to \text{Pos}\,(\mathcal{X}_A \otimes \mathcal{X}_B)$ must have the form

$$\mu(a) = \sum_{b \in \Sigma} P_{a,b} \otimes Q_{a,b}$$

for some finite set Σ and two collections

$$\{P_{a,b} \,:\, a \in \Gamma,\, b \in \Sigma\} \subset \text{Pos}\,(\mathcal{X}_A)\,,$$
$$\{Q_{a,b} \,:\, a \in \Gamma,\, b \in \Sigma\} \subset \text{Pos}\,(\mathcal{X}_B)\,.$$

Choose sufficiently small positive real numbers $\alpha, \beta > 0$ such that

$$\alpha \sum_{a,b} P_{a,b} \leq \mathbb{1}_{\mathcal{X}_A} \qquad \text{and} \qquad \beta \sum_{a,b} Q_{a,b} \leq \mathbb{1}_{\mathcal{X}_B},$$

and define a measurement $\nu_A : (\Gamma \times \Sigma) \cup \{\text{fail}\} \to \text{Pos}\,(\mathcal{X}_A)$ as

$$\nu_A(a,b) = \alpha P_{a,b} \qquad \text{and} \qquad \nu(\text{fail}) = \mathbb{1}_{\mathcal{X}_A} - \alpha \sum_{a,b} P_{a,b},$$

and a measurement $\nu_B : (\Gamma \times \Sigma) \cup \{\text{fail}\} \to \text{Pos}\,(\mathcal{X}_B)$ as

$$\nu_B(a,b) = \beta Q_{a,b} \qquad \text{and} \qquad \nu(\text{fail}) = \mathbb{1}_{\mathcal{X}_B} - \beta \sum_{a,b} Q_{a,b}.$$

Now, consider the situation in which Alice performs ν_A and Bob independently performs ν_B. Let us suppose that Bob sends Alice his measurement outcome, and Alice compares this result with her own to determine the final result. If Bob's measurement outcome is "fail," or if Alice's measurement outcome is not equal to Bob's, Alice outputs "fail." If, on the other hand, Alice and Bob obtain the same measurement outcome $(a, b) \in \Gamma \times \Sigma$, Alice outputs a. The measurement ν that they implement is described by

$$\nu(a) = \sum_{b \in \Sigma} \nu_A(a, b) \otimes \nu_B(a, b) = \alpha\beta \sum_{b \in \Sigma} P_{a,b} \otimes Q_{a,b} = \alpha\beta\mu(a)$$

and

$$\nu(\text{fail}) = \mathbb{1}_{\mathcal{X}_A} \otimes \mathbb{1}_{\mathcal{X}_B} - \alpha\beta \sum_{a \in \Gamma} \mu(a).$$

Taking $\gamma = \alpha\beta$ completes the proof. □

It is the case that certain measurements are not separable, and therefore cannot be performed by means of local operations and classical communication. For instance, no LOCC measurement can perfectly distinguish any fixed entangled pure state from all orthogonal states, given that one of the required measurement operators would then necessarily be non-separable. This fact trivially implies that Alice and Bob cannot perform a measurement with respect to any orthonormal basis

$$\{u_a : a \in \Gamma\} \subset \mathcal{X}_A \otimes \mathcal{X}_B$$

of $\mathcal{X}_A \otimes \mathcal{X}_B$ unless that basis consists entirely of product vectors.

Another example along these lines is that Alice and Bob cannot perfectly distinguish symmetric and antisymmetric states of $\mathbb{C}^{\mathbb{Z}_n} \otimes \mathbb{C}^{\mathbb{Z}_n}$ by means of an LOCC measurement. Such a measurement is described by the two-outcome projective measurement $\{R_n, S_n\}$, where

$$R_n = \frac{1}{2}(\mathbb{1} - W_n) \qquad \text{and} \qquad S_n = \frac{1}{2}(\mathbb{1} + W_n),$$

for W_n denoting the swap operator (as was discussed in the previous lecture). The fact that R_n is not separable follows from the fact that it is not PPT:

$$(T \otimes \mathbb{1})(R_n) = -\frac{n-1}{2} P_n + \frac{1}{2} Q_n,$$

where P_n and Q_n are as defined in the previous lecture.

19.2 Impossibility of LOCC distinguishing some sets of states

When we force measurements to have non-separable measurement operators, it is clear that the measurements cannot be performed using local operations and classical communication. Sometimes, however, we may be interested in a task that potentially allows for many different implementations as a measurement.

One interesting scenario along these lines is the task of distinguishing certain sets of pure states. Specifically, suppose that $\mathcal{X}_A$ and $\mathcal{X}_B$ are complex Euclidean spaces and $\{u_1, \ldots, u_k\} \subset \mathcal{X}_A \otimes \mathcal{X}_B$ is a set of orthogonal unit vectors. Alice and Bob are given a pure state u_i for $i \in \{1, \ldots, k\}$ and their goal is to determine the value of i. It is assumed that they have complete knowledge of the set $\{u_1, \ldots, u_k\}$. Under the assumption that k is smaller than the total dimension of the space $\mathcal{X}_A \otimes \mathcal{X}_B$, there will be many measurements $\mu : \{1, \ldots, k\} \to \text{Pos}(\mathcal{X}_A \otimes \mathcal{X}_B)$ that correctly distinguish the elements of the set $\{u_1, \ldots, u_k\}$, and the general question we consider is whether there is at least one such measurement that is LOCC.

19.2.1 Sets of maximally entangled states

Let us start with a simple general result that proves that sufficiently many maximally entangled pure states are hard to distinguish in the sense described. Specifically, assume that $\mathcal{X}_A$ and $\mathcal{X}_B$ both have dimension n, and consider a collection $U_1, \ldots, U_k \in \mathrm{U}(\mathcal{X}_B, \mathcal{X}_A)$ of pairwise orthogonal unitary operators, meaning that $\langle U_i, U_j \rangle = 0$ for $i \neq j$. The set

$$\left\{ \frac{1}{\sqrt{n}} \operatorname{vec}(U_1), \ldots, \frac{1}{\sqrt{n}} \operatorname{vec}(U_k) \right\}$$

therefore represents a set of maximally entangled pure states. We will show that such a set cannot be perfectly distinguished by a separable measurement, under the assumption that $k \geq n+1$.

Suppose $\mu : \{1, \ldots, k\} \to \mathrm{Pos}(\mathcal{X}_A \otimes \mathcal{X}_B)$ is a separable measurement, so that

$$\mu(j) = \sum_{i=1}^{m} P_{j,i} \otimes Q_{j,i}$$

for each $j = 1, \ldots, k$, for $\{P_{j,i}\} \subset \mathrm{Pos}(\mathcal{X}_A)$ and $\{Q_{j,i}\} \subset \mathrm{Pos}(\mathcal{X}_B)$ being collections of positive semidefinite operators. It follows that

$$\begin{aligned} \langle \mu(j), \operatorname{vec}(U_j) \operatorname{vec}(U_j)^* \rangle &= \sum_{i=1}^{m} \operatorname{Tr}\left(U_j^* P_{j,i} U_j Q_{j,i}^{\mathsf{T}} \right) \\ &\leq \sum_{i=1}^{m} \operatorname{Tr}\left(U_j^* P_{j,i} U_j \right) \operatorname{Tr}\left(Q_{j,i}^{\mathsf{T}} \right) = \sum_{i=1}^{m} \operatorname{Tr}\left(P_{j,i} \right) \operatorname{Tr}\left(Q_{j,i} \right) = \sum_{i=1}^{m} \operatorname{Tr}\left(P_{j,i} \otimes Q_{j,i} \right) = \operatorname{Tr}(\mu(j)) \end{aligned}$$

for each j (where the inequality holds because $\operatorname{Tr}(AB) \leq \operatorname{Tr}(A)\operatorname{Tr}(B)$ for $A, B \geq 0$). Thus, it holds that

$$\frac{1}{k} \sum_{j=1}^{k} \left\langle \mu(j), \frac{1}{n} \operatorname{vec}(U_j) \operatorname{vec}(U_j)^* \right\rangle \leq \frac{1}{nk} \sum_{j=1}^{k} \operatorname{Tr}(\mu(j)) = \frac{n}{k},$$

which implies that the correctness probability of any separable measurement to distinguish the k maximally entangled states is smaller than 1 for $k \geq n+1$.

Naturally, as any measurement implementable by an LOCC protocol is separable, it follows that no LOCC protocol can distinguish more than n maximally entangled states in $\mathcal{X}_A \otimes \mathcal{X}_B$ in the case that $\dim(\mathcal{X}_A) = \dim(\mathcal{X}_B) = n$.

19.2.2 Indistinguishable sets of product states

It is reasonable to hypothesize that large sets of maximally entangled states are not LOCC distinguishable because they are highly entangled. However, it turns out that entanglement is not an essential feature for this phenomenon. In fact, there exist orthogonal collections of *product states* that are not perfectly distinguishable by LOCC measurements.

One example is the following orthonormal basis of $\mathbb{C}^{\mathbb{Z}_3} \otimes \mathbb{C}^{\mathbb{Z}_3}$:

$$\begin{array}{lllll} |1\rangle \otimes |1\rangle & |0\rangle \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) & |2\rangle \otimes \left(\frac{|1\rangle + |2\rangle}{\sqrt{2}}\right) & \left(\frac{|1\rangle + |2\rangle}{\sqrt{2}}\right) \otimes |0\rangle & \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes |2\rangle \\ & |0\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) & |2\rangle \otimes \left(\frac{|1\rangle - |2\rangle}{\sqrt{2}}\right) & \left(\frac{|1\rangle - |2\rangle}{\sqrt{2}}\right) \otimes |0\rangle & \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \otimes |2\rangle \end{array}$$

A measurement with respect to this basis is an example of a measurement that is separable but not LOCC.

The proof that the above set is not perfectly LOCC distinguishable is technical, and so a reference will have to suffice in place of a proof:

> C. H. Bennett, D. DiVincenzo, C. Fuchs, T. Mor, E. Rains, P. Shor, J. Smolin, and W. Wootters. Quantum nonlocality without entanglement. *Physical Review A*, 59(2):1070–1091, 1999.

Another family of examples of product states (but not product bases) that cannot be distinguished by LOCC measurements comes from an unextendible product set. For instance, the set discussed in the previous lecture cannot be distinguished by an LOCC measurement:

$$|0\rangle \otimes \left(\tfrac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \qquad \left(\tfrac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \otimes |2\rangle \qquad \left(\tfrac{|0\rangle + |1\rangle + |2\rangle}{\sqrt{3}}\right) \otimes \left(\tfrac{|0\rangle + |1\rangle + |2\rangle}{\sqrt{3}}\right)$$

$$|2\rangle \otimes \left(\tfrac{|1\rangle - |2\rangle}{\sqrt{2}}\right) \qquad \left(\tfrac{|1\rangle - |2\rangle}{\sqrt{2}}\right) \otimes |0\rangle$$

This fact is proved in the following paper:

> D. DiVincenzo, T. Mor, P. Shor, J. Smolin and B. Terhal. Unextendible Product Bases, Uncompletable Product Bases and Bound Entanglement. *Communications in Mathematical Physics*, 238(3): 379–410, 2003.

19.3 Any two orthogonal pure states can be distinguished

Finally, we will prove an interesting and fundamental result in this area, which is that any two orthogonal pure states can always be distinguished by an LOCC measurement.

In order to prove this fact, we need a theorem known as the Toeplitz–Hausdorff theorem, which concerns the *numerical range* of an operator. The numerical range of an operator $A \in \mathrm{L}(\mathcal{X})$ is the set $\mathcal{N}(A) \subset \mathbb{C}$ defined as follows:

$$\mathcal{N}(A) = \{u^* A u \,:\, u \in \mathcal{X},\, \|u\| = 1\}.$$

This set is also sometimes called the *field of values* of A. It is not hard to prove that the numerical range of a normal operator is simply the convex hull of its eigenvalues. For non-normal operators, however, this is not the case—but the numerical range will nevertheless be a compact and convex set that includes the eigenvalues. The fact that the numerical range is compact and convex is what is stated by the Toeplitz-Hausdorff theorem.

Theorem 19.3 (The Toeplitz-Hausdorff theorem)**.** *For any complex Euclidean space $\mathcal{X}$ and any operator $A \in \mathrm{L}(\mathcal{X})$, the set $\mathcal{N}(A)$ is compact and convex.*

Proof. The proof of compactness is straightforward. Specifically, the function $f : \mathcal{X} \to \mathbb{C}$ defined by $f(u) = u^* A u$ is continuous, and the unit sphere $\mathrm{S}(\mathcal{X})$ is compact. Continuous functions map compact sets to compact sets, implying that $\mathcal{N}(A) = f(\mathrm{S}(\mathcal{X}))$ is compact.

The proof of convexity is the more difficult part of the proof. Let us fix some arbitrary choice of $\alpha, \beta \in \mathcal{N}(A)$ and $p \in [0,1]$. It is our goal to prove that $p\alpha + (1-p)\beta \in \mathcal{N}(A)$. We will assume that $\alpha \neq \beta$, as the assertion is trivial in case $\alpha = \beta$.

By the definition of the numerical range, we may choose unit vectors $u, v \in \mathcal{X}$ such that $u^* A u = \alpha$ and $v^* A v = \beta$. It follows from the fact that $\alpha \neq \beta$ that the vectors u and v are linearly independent.

Next, define

$$B = \frac{-\beta}{\alpha - \beta}\mathbb{1}_{\mathcal{X}} + \frac{1}{\alpha - \beta}A$$

so that $u^*Bu = 1$ and $v^*Bv = 0$. Let

$$X = \frac{1}{2}(B + B^*) \quad \text{and} \quad Y = \frac{1}{2i}(B - B^*).$$

It holds that $B = X + iY$, and both X and Y are Hermitian. It therefore follows that

$$\begin{aligned} u^*Xu &= 1, \qquad & v^*Xv &= 0, \\ u^*Yu &= 0, \qquad & v^*Yv &= 0. \end{aligned}$$

Without loss of generality we may also assume u^*Yv is purely imaginary (i.e., has real part equal to 0), for otherwise v may be replaced by $e^{i\theta}v$ for an appropriate choice of θ without changing any of the previously observed properties.

As u and v are linearly independent, we have that $tu + (1-t)v$ is a nonzero vector for every choice of t. Thus, for each $t \in [0,1]$ we may define

$$z(t) = \frac{tu + (1-t)v}{\|tu + (1-t)v\|},$$

which is of course a unit vector. Because $u^*Yu = v^*Yv = 0$ and u^*Yv is purely imaginary, we have $z(t)^*Yz(t) = 0$ for every t. Thus

$$z(t)^*Bz(t) = z(t)^*Xz(t) = \frac{t^2 + 2t(1-t)\Re(v^*Xu)}{\|tu + (1-t)v\|}.$$

This is a continuous real-valued function mapping 0 to 0 and 1 to 1. Consequently there must exist some choice of $t \in [0,1]$ such that $z(t)^*Bz(t) = p$. Let $w = z(t)$ for such a value of t, so that $w^*Bw = p$. We have that w is a unit vector, and

$$w^*Aw = (\alpha - \beta)\left(\frac{\beta}{\alpha - \beta} + w^*Bw\right) = \beta + p(\alpha - \beta) = p\alpha + (1-p)\beta.$$

Thus we have shown that $p\alpha + (1-p)\beta \in \mathcal{N}(A)$ as required. □

Corollary 19.4. *For any complex Euclidean space $\mathcal{X}$ and any operator $A \in \mathrm{L}(\mathcal{X})$ satisfying $\mathrm{Tr}(A) = 0$, there exists an orthonormal basis $\{x_1, \ldots, x_n\}$ of $\mathcal{X}$ for which $x_i^*Ax_i = 0$ for $i = 1, \ldots, n$.*

Proof. The proof is by induction on $n = \dim(\mathcal{X})$, and the base case $n = 1$ is trivial.

Suppose that $n \geq 2$. It is clear that $\lambda_1(A), \ldots, \lambda_n(A) \in \mathcal{N}(A)$, and thus $0 \in \mathcal{N}(A)$ because

$$0 = \frac{1}{n}\mathrm{Tr}(A) = \frac{1}{n}\sum_{i=1}^{n}\lambda_i(A),$$

which is a convex combination of elements of $\mathcal{N}(A)$. Therefore there exists a unit vector $u \in \mathcal{X}$ such that $u^*Au = 0$.

Now, let $\mathcal{Y} \subseteq \mathcal{X}$ be the orthogonal complement of u in $\mathcal{X}$, and let $\Pi_{\mathcal{Y}} = \mathbb{1}_{\mathcal{X}} - uu^*$ be the orthogonal projection onto $\mathcal{Y}$. It holds that

$$\mathrm{Tr}(\Pi_{\mathcal{Y}}A\Pi_{\mathcal{Y}}) = \mathrm{Tr}(A) - u^*Au = 0.$$

Moreover, because $\operatorname{im}(\Pi_{\mathcal{Y}} A \Pi_{\mathcal{Y}}) \subseteq \mathcal{Y}$, we may regard $\Pi_{\mathcal{Y}} A \Pi_{\mathcal{Y}}$ as an element of $\mathrm{L}(\mathcal{Y})$. By the induction hypothesis, we therefore have that there exists an orthonormal basis $\{v_1, \ldots, v_{n-1}\}$ of $\mathcal{Y}$ such that $v_i^* \Pi_{\mathcal{Y}} A \Pi_{\mathcal{Y}} v_i = 0$ for $i = 1, \ldots, n-1$. It follows that $\{u, v_1, \ldots, v_{n-1}\}$ is an orthonormal basis of $\mathcal{X}$ with the properties required by the statement of the corollary. □

Now we are ready to return to the problem of distinguishing orthogonal states. Suppose that

$$x, y \in \mathcal{X}_A \otimes \mathcal{X}_B$$

are orthogonal unit vectors. We wish to show that there exists an LOCC measurement that correctly distinguishes between x and y. Let $X, Y \in \mathrm{L}(\mathcal{X}_B, \mathcal{X}_A)$ be operators satisfying $x = \operatorname{vec}(X)$ and $y = \operatorname{vec}(Y)$, so that the orthogonality of x and y is equivalent to $\operatorname{Tr}(X^* Y) = 0$. By Corollary 19.4, we have that there exists an orthonormal basis $\{u_1, \ldots, u_n\}$ of $\mathcal{X}_B$ with the property that $u_i^* X^* Y u_i = 0$ for $i = 1, \ldots, n$.

Now, suppose that Bob measures his part of either xx^* or yy^* with respect to the orthonormal basis $\{\overline{u_1}, \ldots, \overline{u_n}\}$ of $\mathcal{X}_B$ and transmits the result of the measurement to Alice. Conditioned on Bob obtaining the outcome i, the (unnormalized) state of Alice's system becomes

$$(\mathbb{1}_{\mathcal{X}_A} \otimes u_i^{\mathsf{T}}) \operatorname{vec}(X) \operatorname{vec}(X)^* (\mathbb{1}_{\mathcal{X}_A} \otimes \overline{u_i}) = X u_i u_i^* X^*$$

in case the original state was xx^*, and $Y u_i u_i^* Y^*$ in case the original state was yy^*.

A necessary and sufficient condition for Alice to be able to correctly distinguish these two states, given knowledge of i, is that

$$\langle X u_i u_i^* X^*, Y u_i u_i^* Y^* \rangle = 0.$$

This condition is equivalent to $u_i^* X^* Y u_i = 0$ for each $i = 1, \ldots, n$. The basis $\{u_1, \ldots, u_n\}$ was chosen to satisfy this condition, which implies that Alice can correctly distinguish the two possibilities without error.