

Lecture 18: The partial transpose and its relationship to entanglement and distillation

In this lecture we will discuss the partial transpose mapping and its connection to entanglement and distillation. Through this study, we will find that there exist *bound-entangled states*, which are states that are entangled and yet have zero distillable entanglement.

18.1 The partial transpose and separability

Recall the Woronowicz–Horodecki criterion for separability: for complex Euclidean spaces \mathcal{X} and \mathcal{Y} , we have that a given operator $P \in \text{Pos}(\mathcal{X} \otimes \mathcal{Y})$ is separable if and only if

$$(\Phi \otimes \mathbb{1}_{L(\mathcal{Y})})(P) \in \text{Pos}(\mathcal{Y} \otimes \mathcal{Y})$$

for every choice of a positive unital mapping $\Phi \in \mathsf{T}(\mathcal{X}, \mathcal{Y})$. We note, however, that the restriction of the mapping Φ to be both unital and to take the form $\Phi \in \mathsf{T}(\mathcal{X}, \mathcal{Y})$ can be relaxed. Specifically, the Woronowicz–Horodecki criterion implies the truth of the following two facts:

1. If $P \in \text{Pos}(\mathcal{X} \otimes \mathcal{Y})$ is separable, then for every choice of a complex Euclidean space \mathcal{Z} and a positive mapping $\Phi \in \mathsf{T}(\mathcal{X}, \mathcal{Z})$, we have

$$(\Phi \otimes \mathbb{1}_{L(\mathcal{Y})})(P) \in \text{Pos}(\mathcal{Z} \otimes \mathcal{Y}).$$

2. If $P \in \text{Pos}(\mathcal{X} \otimes \mathcal{Y})$ is not separable, there exists a positive mapping $\Phi \in \mathsf{T}(\mathcal{X}, \mathcal{Z})$ that reveals this fact, in the sense that

$$(\Phi \otimes \mathbb{1}_{L(\mathcal{Y})})(P) \notin \text{Pos}(\mathcal{Z} \otimes \mathcal{Y}).$$

Moreover, there exists such a mapping Φ that is unital and for which $\mathcal{Z} = \mathcal{Y}$.

It is clear that the criterion illustrates a connection between separability and positive mappings that are not completely positive, for if $\Phi \in \mathsf{T}(\mathcal{X}, \mathcal{Z})$ is completely positive, then

$$(\Phi \otimes \mathbb{1}_{L(\mathcal{Y})})(P) \in \text{Pos}(\mathcal{Z} \otimes \mathcal{Y})$$

for every completely positive mapping $\Phi \in \mathsf{T}(\mathcal{X}, \mathcal{Z})$, regardless of whether P is separable or not.

Thus far, we have only seen one example of a mapping that is positive but not completely positive: the transpose. Let us recall that the transpose mapping $T \in \mathsf{T}(\mathcal{X})$ on a complex Euclidean space \mathcal{X} is defined as

$$T(X) = X^\top$$

for all $X \in L(\mathcal{X})$. The positivity of T is clear: $X \in \text{Pos}(\mathcal{X})$ if and only if $X^\top \in \text{Pos}(\mathcal{X})$ for every $X \in L(\mathcal{X})$. Assuming that $\mathcal{X} = \mathbb{C}^\Sigma$, we have

$$T(X) = \sum_{a,b \in \Sigma} E_{a,b} X E_{a,b} = \sum_{a,b \in \Sigma} E_{a,b} X E_{b,a}^*$$

for all $X \in L(\mathcal{X})$. The Choi–Jamiołkowski representation of T is

$$J(T) = \sum_{a,b \in \Sigma} E_{b,a} \otimes E_{a,b} = W$$

where $W \in U(\mathcal{X} \otimes \mathcal{X})$ denotes the swap operator. The fact that W is not positive semidefinite shows that T is not completely positive.

When we refer to the *partial transpose*, we mean that the transpose mapping is tensored with the identity mapping on some other space. We will use a similar notation to the partial trace: for given complex Euclidean spaces \mathcal{X} and \mathcal{Y} , we define

$$T_{\mathcal{X}} = T \otimes \mathbb{1}_{L(\mathcal{Y})} \in T(\mathcal{X} \otimes \mathcal{Y}).$$

More generally, the subscript refers to the space on which the transpose is performed.

Given that the transpose is positive, we may conclude the following from the Woronowicz–Horodecki criterion for any choice of $P \in \text{Pos}(\mathcal{X} \otimes \mathcal{Y})$:

1. If P is separable, then $T_{\mathcal{X}}(P)$ is necessarily positive semidefinite.
2. If P is not separable, then $T_{\mathcal{X}}(P)$ might or might not be positive semidefinite, although nothing definitive can be concluded from the criterion.

Another way to view these observations is that they describe a sort of one-sided test for entanglement:

1. If $T_{\mathcal{X}}(P)$ is not positive semidefinite for a given $P \in \text{Pos}(\mathcal{X} \otimes \mathcal{Y})$, then P is definitely not separable.
2. If $T_{\mathcal{X}}(P)$ is positive semidefinite for a given $P \in \text{Pos}(\mathcal{X} \otimes \mathcal{Y})$, then P may or may not be separable.

We have seen a specific example where the transpose indeed does identify entanglement: if Σ is a finite, nonempty set of size n , and we take $\mathcal{X}_A = \mathbb{C}^{\Sigma}$ and $\mathcal{X}_B = \mathbb{C}^{\Sigma}$, then

$$P = \frac{1}{n} \sum_{a,b \in \Sigma} E_{a,b} \otimes E_{a,b} \in D(\mathcal{X}_A \otimes \mathcal{X}_B)$$

is certainly entangled, because

$$T_{\mathcal{X}_A}(P) = \frac{1}{n} W \notin \text{Pos}(\mathcal{X}_A \otimes \mathcal{X}_B).$$

We will soon prove that indeed there do exist entangled operators $P \in \text{Pos}(\mathcal{X}_A \otimes \mathcal{X}_B)$ for which $T_{\mathcal{X}_A}(P) \in \text{Pos}(\mathcal{X}_A \otimes \mathcal{X}_B)$, which means that the partial transpose does not give a simple test for separability. It turns out, however, that the partial transpose does have an interesting connection to entanglement distillation, as we will see later in the lecture.

For the sake of discussing this issue in greater detail, let us consider the following definition. For any choice of complex Euclidean spaces \mathcal{X}_A and \mathcal{X}_B , we define

$$\text{PPT}(\mathcal{X}_A : \mathcal{X}_B) = \{P \in \text{Pos}(\mathcal{X}_A \otimes \mathcal{X}_B) : T_{\mathcal{X}_A}(P) \in \text{Pos}(\mathcal{X}_A \otimes \mathcal{X}_B)\}.$$

The acronym PPT stands for *positive partial transpose*.

It is the case that the set $\text{PPT}(\mathcal{X}_A : \mathcal{X}_B)$ is a closed convex cone. Let us also note that this notion respects tensor products, meaning that if $P \in \text{PPT}(\mathcal{X}_A : \mathcal{X}_B)$ and $Q \in \text{PPT}(\mathcal{Y}_A : \mathcal{Y}_B)$, then

$$P \otimes Q \in \text{PPT}(\mathcal{X}_A \otimes \mathcal{Y}_A : \mathcal{X}_B \otimes \mathcal{Y}_B).$$

Finally, notice that the definition of $\text{PPT}(\mathcal{X}_A : \mathcal{X}_B)$ does not really depend on the fact that the partial transpose is performed on \mathcal{X}_A as opposed to \mathcal{X}_B . This follows from the observation that

$$T(T_{\mathcal{X}_A}(X)) = T_{\mathcal{X}_B}(X)$$

for every $X \in L(\mathcal{X}_A \otimes \mathcal{X}_B)$, and therefore

$$T_{\mathcal{X}_A}(X) \in \text{Pos}(\mathcal{X}_A \otimes \mathcal{X}_B) \Leftrightarrow T_{\mathcal{X}_B}(X) \in \text{Pos}(\mathcal{X}_A \otimes \mathcal{X}_B).$$

18.2 Examples of non-separable PPT operators

In this section we will discuss two examples of operators that are both entangled and PPT. This shows that the partial transpose test does not give an efficient test for separability, and also implies something interesting about entanglement distillation to be discussed in the next section.

18.2.1 First example

Let us begin by considering the following collection of operators, all of which act on the complex Euclidean space $\mathbb{C}^{\mathbb{Z}_n} \otimes \mathbb{C}^{\mathbb{Z}_n}$ for an integer $n \geq 2$. We let

$$W_n = \sum_{a,b \in \mathbb{Z}_n} E_{b,a} \otimes E_{a,b}$$

denote the swap operator, which we have now seen several times. It satisfies $W_n(u \otimes v) = v \otimes u$ for all $u, v \in \mathbb{C}^{\mathbb{Z}_n}$. Let us also define

$$\begin{aligned} P_n &= \frac{1}{n} \sum_{a,b \in \mathbb{Z}_n} E_{a,b} \otimes E_{a,b}, & R_n &= \frac{1}{2} \mathbb{1} \otimes \mathbb{1} - \frac{1}{2} W_n, \\ Q_n &= \mathbb{1} \otimes \mathbb{1} - P_n, & S_n &= \frac{1}{2} \mathbb{1} \otimes \mathbb{1} + \frac{1}{2} W_n. \end{aligned} \tag{18.1}$$

It holds that $P_n, Q_n, R_n,$ and S_n are projection operators with $P_n + Q_n = R_n + S_n = \mathbb{1} \otimes \mathbb{1}$. The operator R_n is the projection onto the *anti-symmetric subspace* of $\mathbb{C}^{\mathbb{Z}_n} \otimes \mathbb{C}^{\mathbb{Z}_n}$ and S_n is the projection onto the *symmetric subspace* of $\mathbb{C}^{\mathbb{Z}_n} \otimes \mathbb{C}^{\mathbb{Z}_n}$.

We have that

$$(T \otimes \mathbb{1})(P_n) = \frac{1}{n} W_n \quad \text{and} \quad (T \otimes \mathbb{1})(\mathbb{1} \otimes \mathbb{1}) = \mathbb{1} \otimes \mathbb{1},$$

from which the following equations follow:

$$\begin{aligned} (T \otimes \mathbb{1})(P_n) &= -\frac{1}{n} R_n + \frac{1}{n} S_n, & (T \otimes \mathbb{1})(R_n) &= -\frac{n-1}{2} P_n + \frac{1}{2} Q_n, \\ (T \otimes \mathbb{1})(Q_n) &= \frac{n+1}{n} R_n + \frac{n-1}{n} S_n, & (T \otimes \mathbb{1})(S_n) &= \frac{n+1}{2} P_n + \frac{1}{2} Q_n. \end{aligned}$$

Now let us suppose we have registers $X_2, Y_2, X_3,$ and $Y_3,$ where

$$\mathcal{X}_2 = \mathbb{C}^{\mathbb{Z}_2}, \quad \mathcal{Y}_2 = \mathbb{C}^{\mathbb{Z}_2}, \quad \mathcal{X}_3 = \mathbb{C}^{\mathbb{Z}_3}, \quad \mathcal{Y}_3 = \mathbb{C}^{\mathbb{Z}_3}.$$

In other words, X_2 and Y_2 are qubit registers, while X_3 and Y_3 are qutrit registers. We will imagine the situation in which Alice holds registers X_2 and $X_3,$ while Bob holds Y_2 and $Y_3.$

For every choice of $\alpha > 0,$ define

$$X_\alpha = Q_3 \otimes Q_2 + \alpha P_3 \otimes P_2 \in \text{Pos}(\mathcal{X}_3 \otimes \mathcal{Y}_3 \otimes \mathcal{X}_2 \otimes \mathcal{Y}_2).$$

Based on the above equations we compute:

$$\begin{aligned} T_{\mathcal{X}_3 \otimes \mathcal{X}_2}(X_\alpha) &= \left(\frac{4}{3}R_3 + \frac{2}{3}S_3 \right) \otimes \left(\frac{3}{2}R_2 + \frac{1}{2}S_2 \right) + \alpha \left(-\frac{1}{3}R_3 + \frac{1}{3}S_3 \right) \otimes \left(-\frac{1}{2}R_2 + \frac{1}{2}S_2 \right) \\ &= \frac{12 + \alpha}{6}R_3 \otimes R_2 + \frac{4 - \alpha}{6}R_3 \otimes S_2 + \frac{6 - \alpha}{6}S_3 \otimes R_2 + \frac{2 + \alpha}{6}S_3 \otimes S_2. \end{aligned}$$

Provided that $\alpha \leq 4,$ we therefore have that $X_\alpha \in \text{PPT}(\mathcal{X}_3 \otimes \mathcal{X}_2 : \mathcal{Y}_3 \otimes \mathcal{Y}_2).$

On the other hand, we have that

$$X_\alpha \notin \text{Sep}(\mathcal{X}_3 \otimes \mathcal{X}_2 : \mathcal{Y}_3 \otimes \mathcal{Y}_2)$$

for every choice of $\alpha > 0,$ as we will now show. Define $\Psi \in \text{T}(\mathcal{X}_2 \otimes \mathcal{Y}_2, \mathcal{X}_3 \otimes \mathcal{Y}_3)$ to be the unique mapping for which $J(\Psi) = X_\alpha.$ Using the identity

$$\Psi(Y) = \text{Tr}_{\mathcal{X}_2 \otimes \mathcal{Y}_2} [J(\Psi) (\mathbb{1} \otimes Y^T)]$$

we see that $\Psi(P_2) = \alpha P_3.$ So, for $\alpha > 0$ we have that Ψ increases min-rank and is therefore not a separable mapping. Thus, it is not the case that X_α is separable.

18.2.2 Unextendible product bases

The second example is based on the notion of an *unextendible product basis.* Although the construction works for any choice of an unextendible product basis, we will just consider one example. Let $\mathcal{X} = \mathbb{C}^{\mathbb{Z}_3}$ and $\mathcal{Y} = \mathbb{C}^{\mathbb{Z}_3},$ and consider the following 5 unit vectors in $\mathcal{X} \otimes \mathcal{Y}:$

$$\begin{aligned} u_1 &= |0\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \\ u_2 &= |2\rangle \otimes \left(\frac{|1\rangle - |2\rangle}{\sqrt{2}} \right) \\ u_3 &= \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \otimes |2\rangle \\ u_4 &= \left(\frac{|1\rangle - |2\rangle}{\sqrt{2}} \right) \otimes |0\rangle \\ u_5 &= \left(\frac{|0\rangle + |1\rangle + |2\rangle}{\sqrt{3}} \right) \otimes \left(\frac{|0\rangle + |1\rangle + |2\rangle}{\sqrt{3}} \right) \end{aligned}$$

There are three relevant facts about this set for the purpose of our discussion:

1. The set $\{u_1, \dots, u_5\}$ is an orthonormal set.
2. Each u_i is a product vector, meaning $u_i = x_i \otimes y_i$ for some choice of $x_1, \dots, x_5 \in \mathcal{X}$ and $y_1, \dots, y_5 \in \mathcal{Y}$.
3. It is impossible to find a sixth non-zero product vector $v \otimes w \in \mathcal{X} \otimes \mathcal{Y}$ that is orthogonal to u_1, \dots, u_5 .

To verify the third property, note that in order for a product vector $v \otimes w$ to be orthogonal to any u_i , it must be that $\langle v, x_i \rangle = 0$ or $\langle w, y_i \rangle = 0$. In order to have $\langle v \otimes w, u_i \rangle = 0$ for $i = 1, \dots, 5$ we must therefore have $\langle v, x_i \rangle = 0$ for at least three distinct choices of i or $\langle w, y_i \rangle = 0$ for at least three distinct choices of i . However, for any three distinct choices of indices $i, j, k \in \{1, \dots, 5\}$ we have $\text{span}\{x_i, x_j, x_k\} = \mathcal{X}$ and $\text{span}\{y_i, y_j, y_k\} = \mathcal{Y}$, which implies that either $v = 0$ or $w = 0$, and therefore $v \otimes w = 0$.

Now, define a projection operator $P \in \text{Pos}(\mathcal{X} \otimes \mathcal{Y})$ as

$$P = \mathbb{1}_{\mathcal{X} \otimes \mathcal{Y}} - \sum_{i=1}^5 u_i u_i^*.$$

Let us first note that $P \in \text{PPT}(\mathcal{X} : \mathcal{Y})$. For each $i = 1, \dots, 5$ we have

$$T_{\mathcal{X}}(u_i u_i^*) = (x_i x_i^*)^{\top} \otimes y_i y_i^* = x_i x_i^* \otimes y_i y_i^* = u_i u_i^*.$$

The second equality follows from the fact that each x_i has only real coefficients, so $x_i = \bar{x}_i$. Thus,

$$T_{\mathcal{X}}(P) = T_{\mathcal{X}}(\mathbb{1}_{\mathcal{X} \otimes \mathcal{Y}}) - \sum_{i=1}^5 T_{\mathcal{X}}(u_i u_i^*) = \mathbb{1}_{\mathcal{X} \otimes \mathcal{Y}} - \sum_{i=1}^5 u_i u_i^* = P \in \text{Pos}(\mathcal{X} \otimes \mathcal{Y}),$$

as claimed.

Now let us assume toward contradiction that P is separable. This implies that it is possible to write

$$P = \sum_{j=1}^m v_j v_j^* \otimes w_j w_j^*$$

for some choice of $v_1, \dots, v_m \in \mathcal{X}$ and $w_1, \dots, w_m \in \mathcal{Y}$. For each $i = 1, \dots, 5$ we have

$$0 = u_i^* P u_i = \sum_{j=1}^m u_i^* (v_j v_j^* \otimes w_j w_j^*) u_i.$$

Therefore, for each $j = 1, \dots, m$ we have $\langle v_j \otimes w_j, u_i \rangle = 0$ for $i = 1, \dots, 5$. This implies that $v_1 \otimes w_1 = \dots = v_m \otimes w_m = 0$, and thus $P = 0$, establishing a contradiction. Consequently P is not separable.

18.3 PPT states and distillation

The last part of this lecture concerns the relationship between the partial transpose and entanglement distillation. Our goal will be to prove that PPT states cannot be distilled, meaning that the distillable entanglement is zero.

Let us begin the discussion with some further properties of PPT states that will be needed. First we will observe that separable mappings respect the positivity of the partial transpose.

Theorem 18.1. Suppose $P \in \text{PPT}(\mathcal{X}_A : \mathcal{X}_B)$ and $\Phi \in \text{SepT}(\mathcal{X}_A, \mathcal{Y}_A : \mathcal{X}_B, \mathcal{Y}_B)$ is a separable mapping. It holds that $\Phi(P) \in \text{PPT}(\mathcal{Y}_A : \mathcal{Y}_B)$.

Proof. Consider any choice of operators $A \in \text{L}(\mathcal{X}_A, \mathcal{Y}_A)$ and $B \in \text{L}(\mathcal{X}_B, \mathcal{Y}_B)$. Given that $P \in \text{PPT}(\mathcal{X}_A : \mathcal{X}_B)$, we have

$$T_{\mathcal{X}_A}(P) \in \text{Pos}(\mathcal{X}_A \otimes \mathcal{X}_B)$$

and therefore

$$(\mathbb{1}_{\mathcal{X}_A} \otimes B)T_{\mathcal{X}_A}(P)(\mathbb{1}_{\mathcal{X}_A} \otimes B^*) \in \text{Pos}(\mathcal{X}_A \otimes \mathcal{Y}_B).$$

The partial transpose on \mathcal{X}_A commutes with the conjugation by B , and therefore

$$T_{\mathcal{X}_A}((\mathbb{1}_{\mathcal{X}_A} \otimes B)P(\mathbb{1}_{\mathcal{X}_A} \otimes B^*)) \in \text{Pos}(\mathcal{X}_A \otimes \mathcal{Y}_B).$$

This implies that

$$T(T_{\mathcal{X}_A}((\mathbb{1} \otimes B)P(\mathbb{1} \otimes B^*))) = T_{\mathcal{Y}_B}((\mathbb{1} \otimes B)P(\mathbb{1} \otimes B^*)) \in \text{Pos}(\mathcal{X}_A \otimes \mathcal{Y}_B)$$

as remarked in the first section of the lecture. Using the fact that conjugation by A commutes with the partial transpose on \mathcal{Y}_B , we have that

$$(A \otimes \mathbb{1}_{\mathcal{Y}_B})T_{\mathcal{Y}_B}((\mathbb{1} \otimes B)P(\mathbb{1} \otimes B^*))(A^* \otimes \mathbb{1}_{\mathcal{Y}_B}) = T_{\mathcal{Y}_B}((A \otimes B)P(A^* \otimes B^*)) \in \text{Pos}(\mathcal{Y}_A \otimes \mathcal{Y}_B).$$

We have therefore proved that $(A \otimes B)P(A^* \otimes B^*) \in \text{PPT}(\mathcal{Y}_A : \mathcal{Y}_B)$.

Now, for $\Phi \in \text{SepT}(\mathcal{X}_A, \mathcal{Y}_A : \mathcal{X}_B, \mathcal{Y}_B)$, we have that $\Phi(P) \in \text{PPT}(\mathcal{Y}_A : \mathcal{Y}_B)$ by the above observation together with the fact that $\text{PPT}(\mathcal{Y}_A : \mathcal{Y}_B)$ is a convex cone. \square

Next, let us note that PPT states cannot have a large inner product with a maximally entangled states.

Lemma 18.2. Let \mathcal{X} and \mathcal{Y} be complex Euclidean spaces and let $n = \min\{\dim(\mathcal{X}), \dim(\mathcal{Y})\}$. For any PPT density operator

$$\rho \in \text{D}(\mathcal{X} \otimes \mathcal{Y}) \cap \text{PPT}(\mathcal{X} : \mathcal{Y})$$

we have $M(\rho) \leq 1/n$.

Proof. Let us assume, without loss of generality, that $\mathcal{Y} = \mathbb{C}^{\mathbb{Z}_n}$ and $\dim(\mathcal{X}) \geq n$. Every maximally entangled state on $\mathcal{X} \otimes \mathcal{Y}$ may therefore be written

$$(U \otimes \mathbb{1}_{\mathcal{Y}})P_n(U \otimes \mathbb{1}_{\mathcal{Y}})^*$$

for $U \in \text{U}(\mathcal{Y}, \mathcal{X})$ being a linear isometry, and where P_n is as defined in (18.1). We have that

$$\langle (U \otimes \mathbb{1}_{\mathcal{Y}})P_n(U \otimes \mathbb{1}_{\mathcal{Y}})^*, \rho \rangle = \langle P_n, (U \otimes \mathbb{1}_{\mathcal{Y}})^*\rho(U \otimes \mathbb{1}_{\mathcal{Y}}) \rangle,$$

and that

$$(U \otimes \mathbb{1}_{\mathcal{Y}})^*\rho(U \otimes \mathbb{1}_{\mathcal{Y}}) \in \text{PPT}(\mathcal{Y} : \mathcal{Y})$$

is a PPT operator with trace at most 1. To prove the lemma it therefore suffices to prove that

$$\langle P_n, \xi \rangle \leq \frac{1}{n}$$

for every $\xi \in \text{D}(\mathcal{Y} \otimes \mathcal{Y}) \cap \text{PPT}(\mathcal{Y} : \mathcal{Y})$.

The partial transpose is its own adjoint and inverse, which implies that

$$\langle (T \otimes \mathbb{1})(A), (T \otimes \mathbb{1})(B) \rangle = \langle A, B \rangle$$

for any choice of operators $A, B \in L(\mathcal{Y} \otimes \mathcal{Y})$. It is also clear that the partial transpose preserves trace, which implies that $(T \otimes \mathbb{1})(\xi) \in D(\mathcal{Y} \otimes \mathcal{Y})$ for every $\xi \in D(\mathcal{Y} \otimes \mathcal{Y}) \cap \text{PPT}(\mathcal{Y} : \mathcal{Y})$. Consequently we have

$$\langle P_n, \xi \rangle = |\langle P_n, \xi \rangle| = |\langle (T \otimes \mathbb{1})(P_n), (T \otimes \mathbb{1})(\xi) \rangle| = \frac{1}{n} |\langle W_n, (T \otimes \mathbb{1})(\xi) \rangle| \leq \frac{1}{n} \|(T \otimes \mathbb{1})(\xi)\|_1 = \frac{1}{n},$$

where the inequality follows from the fact that W_n is unitary and the last equality follows from the fact that $(T \otimes \mathbb{1})(\xi)$ is a density operator. \square

Finally we are ready for the main result of the section, which states that PPT density operators have no distillable entanglement.

Theorem 18.3. *Let \mathcal{X}_A and \mathcal{X}_B be complex Euclidean spaces and let $\rho \in D(\mathcal{X}_A \otimes \mathcal{X}_B) \cap \text{PPT}(\mathcal{X}_A : \mathcal{X}_B)$. It holds that $E_d(\rho) = 0$.*

Proof. Let $\mathcal{Y}_A = \mathbb{C}^{\{0,1\}}$ and $\mathcal{Y}_B = \mathbb{C}^{\{0,1\}}$ be complex Euclidean spaces each corresponding to a single qubit, as in the definition of distillable entanglement, and let $\tau \in D(\mathcal{Y}_A \otimes \mathcal{Y}_B)$ be the density operator corresponding to a perfect e-bit.

Let $\alpha > 0$ and let

$$\Phi_n \in \text{LOCC} \left(\mathcal{X}_A^{\otimes n}, \mathcal{Y}_A^{\otimes \lfloor \alpha n \rfloor} : \mathcal{X}_B^{\otimes n}, \mathcal{Y}_B^{\otimes \lfloor \alpha n \rfloor} \right)$$

be an LOCC channel for each $n \geq 1$. This implies that Φ_n is a separable channel. Now, if $\rho \in \text{PPT}(\mathcal{X}_A : \mathcal{X}_B)$ then $\rho^{\otimes n} \in \text{PPT}(\mathcal{X}_A^{\otimes n} : \mathcal{X}_B^{\otimes n})$, and therefore

$$\Phi_n(\rho^{\otimes n}) \in D \left(\mathcal{Y}_A^{\otimes \lfloor \alpha n \rfloor} \otimes \mathcal{Y}_B^{\otimes \lfloor \alpha n \rfloor} \right) \cap \text{PPT} \left(\mathcal{Y}_A^{\otimes \lfloor \alpha n \rfloor} : \mathcal{Y}_B^{\otimes \lfloor \alpha n \rfloor} \right).$$

By Lemma 18.2 we therefore have that

$$\left\langle \tau^{\otimes \lfloor \alpha n \rfloor}, \Phi_n(\rho^{\otimes n}) \right\rangle \leq 2^{-\lfloor \alpha n \rfloor}.$$

As we have assumed $\alpha > 0$, this implies that

$$\lim_{n \rightarrow \infty} F \left(\Phi_n(\rho^{\otimes n}), \tau^{\otimes \lfloor \alpha n \rfloor} \right) = 0.$$

It follows that $E_d(\rho) < \alpha$, and from this we conclude that $E_d(\rho) = 0$. \square