Lecture 5: Naimark's theorem; characterizations of channels

5.1 Naimark's Theorem

The following theorem expresses a fundamental relationship between ordinary measurements and projective measurements. It is known as Naimark's theorem, although one should understand that it is really a simple, finite-dimensional case of a more general theorem known by the same name that is important in operator theory. The theorem is also sometimes called *Neumark's Theorem*: the two names refer to the same individual, Mark Naimark, whose name has been transliterated in these two different ways.

Theorem 5.1 (Naimark's theorem)**.** *Let $\mathcal{X}$ be a complex Euclidean space, let*

$$\mu : \Gamma \rightarrow \mathrm{Pos}(\mathcal{X})$$

be a measurement on $\mathcal{X}$, and let $\mathcal{Y} = \mathbb{C}^{\Gamma}$. There exists a linear isometry $A \in \mathrm{U}(\mathcal{X}, \mathcal{X} \otimes \mathcal{Y})$ such that

$$\mu(a) = A^*(\mathbb{1}_{\mathcal{X}} \otimes E_{a,a})A$$

for every $a \in \Gamma$.

Proof. Define $A \in \mathrm{L}(\mathcal{X}, \mathcal{X} \otimes \mathcal{Y})$ as

$$A = \sum_{a \in \Gamma} \sqrt{\mu(a)} \otimes e_a.$$

It holds that

$$A^* A = \sum_{a \in \Sigma} \mu(a) = \mathbb{1}_{\mathcal{X}},$$

so A is a linear isometry, and $A^*(\mathbb{1}_{\mathcal{X}} \otimes E_{a,a})A = \mu(a)$ for each $a \in \Gamma$ as required. □

One may interpret this theorem as saying that an arbitrary measurement

$$\mu : \Gamma \rightarrow \mathrm{Pos}(\mathcal{X})$$

on a register X can be simulated by a projective measurement on a pair of registers (X, Y). In particular, we take $\mathcal{Y} = \mathbb{C}^{\Gamma}$, and by the theorem we conclude that there exists a linear isometry $A \in \mathrm{U}(\mathcal{X}, \mathcal{X} \otimes \mathcal{Y})$ such that

$$\mu(a) = A^*(\mathbb{1}_{\mathcal{X}} \otimes E_{a,a})A$$

for every $a \in \Gamma$. For an arbitrary, but fixed, choice of a unit vector $u \in \mathcal{Y}$, we may choose a unitary operator $U \in \mathrm{U}(\mathcal{X} \otimes \mathcal{Y})$ for which

$$A = U(\mathbb{1}_{\mathcal{X}} \otimes u).$$

Consider the projective measurement $\nu : \Gamma \to \mathrm{Pos}(\mathcal{X} \otimes \mathcal{Y})$ defined as

$$\nu(a) = U^*(\mathbb{1}_{\mathcal{X}} \otimes E_{a,a})U$$

for each $a \in \Gamma$. If a density operator $\rho \in \mathrm{D}(\mathcal{X})$ is given, and the registers (X, Y) are prepared in the state $\rho \otimes uu^*$, then this projective measurement results in each outcome $a \in \Gamma$ with probability

$$\langle \nu(a), \rho \otimes uu^* \rangle = \langle A^*(\mathbb{1}_{\mathcal{X}} \otimes E_{a,a})A, \rho \rangle = \langle \mu(a), \rho \rangle .$$

The probability for each measurement outcome is therefore in agreement with the original measurement μ.

5.2 Representations of quantum channels

Next, we will move on to a discussion of linear mappings of the form $\Phi : \mathrm{L}(\mathcal{X}) \to \mathrm{L}(\mathcal{Y})$. Recall that we write $\mathrm{T}(\mathcal{X}, \mathcal{Y})$ to denote the space of all linear mappings taking this form. Mappings of this sort are important in quantum information theory because (among other reasons) channels take this form.

In this section we will discuss four different ways to represent such mappings, as well as a relationship among these representations. Throughout this section, and the remainder of the lecture, we assume that $\mathcal{X} = \mathbb{C}^\Sigma$ and $\mathcal{Y} = \mathbb{C}^\Gamma$ are arbitrary fixed complex Euclidean spaces.

5.2.1 The natural representation

For every mapping $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$, it is clear that the mapping

$$\mathrm{vec}(X) \mapsto \mathrm{vec}(\Phi(X))$$

is linear, as it can be represented as a composition of linear mappings. To be precise, there must exist a linear operator $K(\Phi) \in \mathrm{L}(\mathcal{X} \otimes \mathcal{X}, \mathcal{Y} \otimes \mathcal{Y})$ that satisfies

$$K(\Phi)\,\mathrm{vec}(X) = \mathrm{vec}(\Phi(X))$$

for all $X \in \mathrm{L}(\mathcal{X})$. The operator $K(\Phi)$ is the *natural representation* of Φ. A concrete expression for $K(\Phi)$ is as follows:

$$K(\Phi) = \sum_{a,b \in \Sigma} \sum_{c,d \in \Gamma} \langle E_{c,d}, \Phi(E_{a,b}) \rangle \, E_{c,a} \otimes E_{d,b}. \tag{5.1}$$

To verify that the above equation (5.1) holds, one may simply compute:

$$K(\Phi)\,\mathrm{vec}(E_{a,b}) = \mathrm{vec}\left(\sum_{c,d \in \Gamma} \langle E_{c,d}, \Phi(E_{a,b}) \rangle \, E_{c,d} \right) = \mathrm{vec}(\Phi(E_{a,b})),$$

from which it follows that $K(\Phi)\,\mathrm{vec}(X) = \mathrm{vec}(\Phi(X))$ for all X by linearity.

Notice that the mapping $K : \mathrm{T}(\mathcal{X}, \mathcal{Y}) \to \mathrm{L}(\mathcal{X} \otimes \mathcal{X}, \mathcal{Y} \otimes \mathcal{Y})$ is itself linear:

$$K(\alpha\Phi + \beta\Psi) = \alpha K(\Phi) + \beta K(\Psi)$$

for all choices of $\alpha, \beta \in \mathbb{C}$ and $\Phi, \Psi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$. In fact, K is a linear bijection, as the above form (5.1) makes clear that for every operator $A \in \mathrm{L}(\mathcal{X} \otimes \mathcal{X}, \mathcal{Y} \otimes \mathcal{Y})$ there exists a unique choice of $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ for which $A = K(\Phi)$.

The natural representation respects the notion of adjoints, meaning that

$$K(\Phi^*) = (K(\Phi))^*$$

for every mapping $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$.

5.2.2 The Choi-Jamiołkowski representation

The natural representation of mappings specifies a straightforward way that an operator can be associated with a mapping. There is a different and somewhat less straightforward way that such an association can be made that turns out to be very important in understanding completely positive mappings in particular. Specifically, let us define a mapping $J : \mathrm{T}(\mathcal{X}, \mathcal{Y}) \rightarrow \mathrm{L}(\mathcal{Y} \otimes \mathcal{X})$ as

$$J(\Phi) = \sum_{a,b \in \Sigma} \Phi(E_{a,b}) \otimes E_{a,b}$$

for each $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$. Alternately we may write

$$J(\Phi) = (\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X})}) (\mathrm{vec}(\mathbb{1}_{\mathcal{X}}) \, \mathrm{vec}(\mathbb{1}_{\mathcal{X}})^*) \, .$$

The operator $J(\Phi)$ is called the *Choi-Jamiołkowski representation* of Φ.

As for the natural representation, it is evident from the definition that J is a linear bijection. Another way to see this is to note that the action of the mapping Φ can be recovered from the operator $J(\Phi)$ by means of the equation

$$\Phi(X) = \mathrm{Tr}_{\mathcal{X}} \left[J(\Phi) \, (\mathbb{1}_{\mathcal{Y}} \otimes X^{\mathsf{T}}) \right] .$$

5.2.3 Kraus representations

Let $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ be a mapping, and suppose that

$$\{A_a \, : \, a \in \Gamma\}, \; \{B_a \, : \, a \in \Gamma\} \subset \mathrm{L}(\mathcal{X}, \mathcal{Y})$$

are (finite and nonempty) collections of operators for which the equation

$$\Phi(X) = \sum_{a \in \Gamma} A_a X B_a^* \tag{5.2}$$

holds for all $X \in \mathrm{L}(\mathcal{X})$. The expression (5.2) is said to be a *Kraus representation* of the mapping Φ. It will be established shortly that Kraus representations exist for all mappings. Unlike the natural representation and Choi-Jamiołkowski representation, Kraus representations are never unique.

The term *Kraus representation* is sometimes reserved for the case that $A_a = B_a$ for each $a \in \Gamma$, and in this case the operators $\{A_a \, : \, a \in \Gamma\}$ are called *Kraus operators*. Such a representation exists, as we will see shortly, if and only if Φ is completely positive.

Under the assumption that Φ is given by the above equation (5.2), it holds that

$$\Phi^*(Y) = \sum_{a \in \Gamma} A_a^* Y B_a.$$

5.2.4 Stinespring representations

Finally, suppose that $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ is a given mapping, $\mathcal{Z}$ is a complex Euclidean space, and $A, B \in \mathrm{L}(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$ are operators such that

$$\Phi(X) = \mathrm{Tr}_{\mathcal{Z}} (A X B^*) \tag{5.3}$$

for all $X \in \mathrm{L}(\mathcal{X})$. The expression (5.3) is said to be a *Stinespring representation* of Φ. Similar to Kraus representations, Stinespring representations always exists for a given Φ and are never unique.

Similar to Kraus representations, the term *Stinespring representation* is often reserved for the case $A = B$. Once again, as we will see, such a representation exists if and only if Φ is completely positive.

If $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ is given by the above equation (5.3), then

$$\Phi^*(Y) = A^*(Y \otimes \mathbb{1}_{\mathcal{Z}})B.$$

(Expressions of this form are also sometimes referred to as Stinespring representations.)

5.2.5 A simple relationship among the representations

The following proposition explains a simple way in which the four representations discussed above relate to one another.

Proposition 5.2. *Let $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ and let $\{A_a : a \in \Gamma\}$ and $\{B_a : a \in \Gamma\}$ be collections of operators in $\mathrm{L}(\mathcal{X}, \mathcal{Y})$ for a finite, non-empty set Γ. The following statements are equivalent.*

1. *(Kraus representations.) It holds that*

$$\Phi(X) = \sum_{a \in \Gamma} A_a X B_a^*$$

for all $X \in \mathrm{L}(\mathcal{X})$.

2. *(Stinespring representations.) For $\mathcal{Z} = \mathbb{C}^\Gamma$ and $A, B \in \mathrm{L}(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$ defined as*

$$A = \sum_{a \in \Gamma} A_a \otimes e_a \qquad \text{and} \qquad B = \sum_{a \in \Gamma} B_a \otimes e_a,$$

it holds that $\Phi(X) = \mathrm{Tr}_{\mathcal{Z}}(AXB^*)$ for all $X \in \mathrm{L}(\mathcal{X})$.

3. *(The natural representation.) It holds that*

$$K(\Phi) = \sum_{a \in \Gamma} A_a \otimes \overline{B_a}.$$

4. *(The Choi-Jamiołkowski representation.) It holds that*

$$J(\Phi) = \sum_{a \in \Gamma} \operatorname{vec}(A_a) \operatorname{vec}(B_a)^*.$$

Proof. The equivalence between items 1 and 2 is a straightforward calculation. The equivalence between items 1 and 3 follows from the identity

$$\operatorname{vec}(A_a X B_a^*) = \left(A_a \otimes \overline{B_a}\right) \operatorname{vec}(X)$$

for each $a \in \Gamma$ and every $X \in \mathrm{L}(\mathcal{X})$. Finally, the equivalence between items 1 and 4 follows from the expression

$$J(\Phi) = (\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X})}) \left(\operatorname{vec}(\mathbb{1}_{\mathcal{X}}) \operatorname{vec}(\mathbb{1}_{\mathcal{X}})^*\right)$$

along with

$$(A_a \otimes \mathbb{1}_{\mathcal{X}}) \operatorname{vec}(\mathbb{1}_{\mathcal{X}}) = \operatorname{vec}(A_a) \quad \text{and} \quad \operatorname{vec}(\mathbb{1}_{\mathcal{X}})^* (B_a^* \otimes \mathbb{1}_{\mathcal{X}}) = \operatorname{vec}(B_a)^*$$

for each $a \in \Gamma$. □

Various facts may be derived from the above proposition. For instance, it follows that every mapping $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ has a Kraus representation in which $|\Gamma| = \operatorname{rank}(J(\Phi)) \leq \dim(\mathcal{X} \otimes \mathcal{Y})$, and similarly that every such Φ has a Stinespring representation in which $\dim(\mathcal{Z}) = \operatorname{rank}(J(\Phi))$.

5.3 Characterizations of completely positive and trace-preserving maps

Now we are ready to characterize quantum channels in terms of their Choi-Jamiołkowski, Kraus, and Stinespring representations. (The natural representation does not happen to help us with respect to these particular characterizations—which is not surprising because it essentially throws away the operator structure of the inputs and outputs of a given mapping.)

5.3.1 Characterizations of completely positive maps

We will begin with a characterization of completely positive mappings in terms of their Choi-Jamiołkowski, Kraus, and Stinespring representations. Before doing this, let us recall the following terminology: a mapping $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ is said to be *positive* if and only if $\Phi(P) \in \mathrm{Pos}(\mathcal{Y})$ for all $P \in \mathrm{Pos}(\mathcal{X})$, and is said to be *completely positive* if and only if $\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{Z})}$ is positive for every choice of a complex Euclidean space $\mathcal{Z}$.

Theorem 5.3. *For every mapping $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$, the following statements are equivalent.*

1. *Φ is completely positive.*
2. *$\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X})}$ is positive.*
3. *$J(\Phi) \in \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{X})$.*
4. *There exists a finite set of operators $\{A_a : a \in \Gamma\} \subset \mathrm{L}(\mathcal{X}, \mathcal{Y})$ such that*

$$\Phi(X) = \sum_{a \in \Gamma} A_a X A_a^* \tag{5.4}$$

   for all $X \in \mathrm{L}(\mathcal{X})$.

5. *Item 4 holds for $|\Gamma| = \mathrm{rank}(J(\Phi))$.*
6. *There exists a complex Euclidean space $\mathcal{Z}$ and an operator $A \in \mathrm{L}(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$ such that*

$$\Phi(X) = \mathrm{Tr}_{\mathcal{Z}}(AXA^*)$$

   for all $X \in \mathrm{L}(\mathcal{X})$.

7. *Item 6 holds for $\mathcal{Z}$ having dimension equal to the rank of $J(\Phi)$.*

Proof. The theorem will be proved by establishing implications among the 7 items that are sufficient to establish their equivalence. The particular implications that will be proved are summarized as follows:

$$\begin{gathered} (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (5) \Rightarrow (4) \Rightarrow (1) \\ (5) \Rightarrow (7) \Rightarrow (6) \Rightarrow (1) \end{gathered}$$

Note that some of these implications are immediate: item 1 implies item 2 by the definition of complete positivity, item 5 trivially implies item 4, item 7 trivially implies item 6, and item 5 implies item 7 by Proposition 5.2.

Assume that $\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X})}$ is positive. Given that

$$\mathrm{vec}(\mathbb{1}_{\mathcal{X}})\,\mathrm{vec}(\mathbb{1}_{\mathcal{X}})^* \in \mathrm{Pos}(\mathcal{X} \otimes \mathcal{X})$$

and

$$J(\Phi) = \left(\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X})}\right)(\mathrm{vec}(\mathbb{1}_{\mathcal{X}})\,\mathrm{vec}(\mathbb{1}_{\mathcal{X}})^*),$$

it follows that $J(\Phi) \in \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{X})$. Item 2 therefore implies item 3.

Next, assume that $J(\Phi) \in \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{X})$. By the spectral theorem, along with the fact that every eigenvalue of a positive semidefinite operator is non-negative, we have that it is possible to write

$$J(\Phi) = \sum_{a \in \Gamma} u_a u_a^*,$$

for some choice of vectors $\{u_a : a \in \Gamma\} \subset \mathcal{Y} \otimes \mathcal{X}$ such that $|\Gamma| = \mathrm{rank}(J(\Phi))$. Defining $A_a \in \mathrm{L}(\mathcal{X}, \mathcal{Y})$ so that $\mathrm{vec}(A_a) = u_a$ for each $a \in \Gamma$, it follows that

$$J(\Phi) = \sum_{a \in \Gamma} \mathrm{vec}(A_a) \,\mathrm{vec}(A_a)^*.$$

The equation (5.4) therefore holds for every $X \in \mathrm{L}(\mathcal{X})$ by Proposition 5.2, which establishes that item 3 implies item 5.

Finally, note that mappings of the form $X \mapsto AXA^*$ are easily seen to be completely positive, and non-negative linear combinations of completely positive mappings are completely positive as well. Item 4 therefore implies item 1. Along similar lines, the partial trace is completely positive and completely positive mappings are closed under composition. Item 6 therefore implies item 1, which completes the proof. □

5.3.2 Characterizations of trace-preserving maps

Next we will characterize the collection of trace-preserving mappings in terms of their representations. These characterizations are straightforward, but it is nevertheless convenient to state them in a similar style to those of Theorem 5.3.

Before stating the theorem, the following terminology must be mentioned. A mapping $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ is said to be *unital* if and only if $\Phi(\mathbb{1}_{\mathcal{X}}) = \mathbb{1}_{\mathcal{Y}}$.

Theorem 5.4. *For every mapping $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$, the following statements are equivalent.*

1. *Φ is trace-preserving.*
2. *Φ^* is unital.*
3. $\mathrm{Tr}_{\mathcal{Y}}(J(\Phi)) = \mathbb{1}_{\mathcal{X}}$.
4. *There exists a Kraus representation*

   $$\Phi(X) = \sum_{a \in \Gamma} A_a X B_a^*$$

   of Φ for which the operators $\{A_a : a \in \Gamma\}, \{B_a : a \in \Gamma\} \subset \mathrm{L}(\mathcal{X}, \mathcal{Y})$ satisfy

   $$\sum_{a \in \Gamma} A_a^* B_a = \mathbb{1}_{\mathcal{X}}.$$

5. *For all Kraus representations*

   $$\Phi(X) = \sum_{a \in \Gamma} A_a X B_a^*$$

   of Φ, the operators $\{A_a : a \in \Gamma\}, \{B_a : a \in \Gamma\} \subset \mathrm{L}(\mathcal{X}, \mathcal{Y})$ satisfy

   $$\sum_{a \in \Gamma} A_a^* B_a = \mathbb{1}_{\mathcal{X}}.$$

6. *There exists a Stinespring representation*

$$\Phi(X) = \operatorname{Tr}_{\mathcal{Z}}(AXB^*)$$

of Φ *for which the operators* $A, B \in \mathrm{L}(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$ *satisfy* $A^*B = \mathbb{1}_{\mathcal{X}}$.

7. *For all Stinespring representations*

$$\Phi(X) = \operatorname{Tr}_{\mathcal{Z}}(AXB^*)$$

of Φ, *the operators* $A, B \in \mathrm{L}(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$ *satisfy* $A^*B = \mathbb{1}_{\mathcal{X}}$.

Proof. Under the assumption that Φ is trace-preserving, it holds that

$$\langle \mathbb{1}_{\mathcal{X}}, X \rangle = \operatorname{Tr}(X) = \operatorname{Tr}(\Phi(X)) = \langle \mathbb{1}_{\mathcal{Y}}, \Phi(X) \rangle = \langle \Phi^*(\mathbb{1}_{\mathcal{Y}}), X \rangle,$$

and thus $\langle \mathbb{1}_{\mathcal{X}} - \Phi^*(\mathbb{1}_{\mathcal{Y}}), X \rangle = 0$ for all $X \in \mathrm{L}(\mathcal{X})$. It follows that $\Phi^*(\mathbb{1}_{\mathcal{Y}}) = \mathbb{1}_{\mathcal{X}}$, so Φ^* is unital. Along similar lines, the assumption that Φ^* is unital implies that

$$\operatorname{Tr}(\Phi(X)) = \langle \mathbb{1}_{\mathcal{Y}}, \Phi(X) \rangle = \langle \Phi^*(\mathbb{1}_{\mathcal{Y}}), X \rangle = \langle \mathbb{1}_{\mathcal{X}}, X \rangle = \operatorname{Tr}(X)$$

for every $X \in \mathrm{L}(\mathcal{X})$, so Φ is trace-preserving. The equivalence of items 1 and 2 has therefore been established.

Next, suppose that

$$\Phi(X) = \sum_{a \in \Gamma} A_a X B_a^*$$

is a Kraus representation of Φ. It holds that

$$\Phi^*(Y) = \sum_{a \in \Gamma} A_a^* Y B_a$$

for every $Y \in \mathrm{L}(\mathcal{Y})$, and in particular it holds that

$$\Phi^*(\mathbb{1}_{\mathcal{Y}}) = \sum_{a \in \Gamma} A_a^* B_a.$$

Thus, if Φ^* is unital, then

$$\sum_{a \in \Gamma} A_a^* B_a = \mathbb{1}_{\mathcal{X}}, \tag{5.5}$$

and so it has been proved that item 2 implies item 5. On the other hand, if (5.5) holds, then it follows that $\Phi^*(\mathbb{1}_{\mathcal{Y}}) = \mathbb{1}_{\mathcal{X}}$, and therefore item 4 implies item 2. Given that every mapping has at least one Kraus representation, item 5 trivially implies item 4, and therefore the equivalence of items 2, 4, and 5 has been established.

Now assume that

$$\Phi(X) = \operatorname{Tr}_{\mathcal{Z}}(AXB^*)$$

is a Stinespring representation of Φ. It follows that

$$\Phi^*(Y) = A^*(Y \otimes \mathbb{1}_{\mathcal{Z}})B$$

for all $Y \in \mathrm{L}(\mathcal{Y})$, and in particular $\Phi^*(\mathbb{1}_{\mathcal{Y}}) = A^*B$. The equivalence of items 2, 6 and 7 follows by the same reasoning as for the case of items 2, 4 and 5.

Finally, suppose that $\mathcal{X} = \mathbb{C}^\Gamma$, and consider the operator

$$\operatorname{Tr}_{\mathcal{Y}}(J(\Phi)) = \sum_{a,b\in\Gamma} \operatorname{Tr}\left(\Phi(E_{a,b})\right) E_{a,b}. \tag{5.6}$$

If it holds that Φ is trace-preserving, then it follows that

$$\operatorname{Tr}(\Phi(E_{a,b})) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b, \end{cases} \tag{5.7}$$

and therefore

$$\operatorname{Tr}_{\mathcal{Y}}(J(\Phi)) = \sum_{a\in\Gamma} E_{a,a} = \mathbb{1}_{\mathcal{X}}.$$

Conversely, if it holds that $\operatorname{Tr}_{\mathcal{Y}}(J(\Phi)) = \mathbb{1}_{\mathcal{X}}$, then by the expression (5.6) it follows that (5.7) holds. The mapping Φ is therefore trace-preserving by linearity and the fact that $\{E_{a,b} : a, b \in \Gamma\}$ is a basis of $\mathrm{L}(\mathcal{X})$. The equivalence of items 1 and 3 has therefore been established, which completes the proof. □

5.3.3 Characterizations of channels

The two theorems from above are now easily combined to give the following characterization of quantum channels. For convenience, let us hereafter write $\mathrm{C}(\mathcal{X}, \mathcal{Y})$ to denote the set of all channels from $\mathcal{X}$ to $\mathcal{Y}$, meaning the set of all completely positive and trace-preserving mappings $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$.

Corollary 5.5. *Let $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$. The following statements are equivalent.*

1. $\Phi \in \mathrm{C}(\mathcal{X}, \mathcal{Y})$.
2. $J(\Phi) \in \operatorname{Pos}(\mathcal{Y} \otimes \mathcal{X})$ *and* $\operatorname{Tr}_{\mathcal{Y}}(J(\Phi)) = \mathbb{1}_{\mathcal{X}}$.
3. *There exists a finite set of operators $\{A_a : a \in \Gamma\} \subset \mathrm{L}(\mathcal{X}, \mathcal{Y})$ such that*

   $$\Phi(X) = \sum_{a\in\Gamma} A_a X A_a^*$$

   for all $X \in \mathrm{L}(\mathcal{X})$, and such that

   $$\sum_{a\in\Gamma} A_a^* A_a = \mathbb{1}_{\mathcal{X}}.$$
4. *Item 3 holds for Γ satisfying $|\Gamma| = \operatorname{rank}(J(\Phi))$.*
5. *There exists a complex Euclidean space $\mathcal{Z}$ and a linear isometry $A \in \mathrm{U}(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$, such that*

   $$\Phi(X) = \operatorname{Tr}_{\mathcal{Z}}(AXA^*)$$

   for all $X \in \mathrm{L}(\mathcal{X})$.
6. *Item 5 holds for a complex Euclidean space $\mathcal{Z}$ with $\dim(\mathcal{Z}) = \operatorname{rank}(J(\Phi))$.*