Time-reversal of rank-one quantum strategy functions

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The *quantum strategy* (or *quantum combs*) framework is a useful tool for reasoning about interactions among entities that process and exchange quantum information over the course of multiple turns. We prove a time-reversal property for a class of linear functions, defined on quantum strategy representations within this framework, that corresponds to the set of rank-one positive semidefinite operators on a certain space. This time-reversal property states that the maximum value obtained by such a function over all valid quantum strategies is also obtained when the direction of time for the function is reversed, despite the fact that the strategies themselves are generally not time reversible. An application of this fact is an alternative proof of a known relationship between the conditional min- and max-entropy of bipartite quantum states, along with generalizations of this relationship.

1 The quantum strategy framework

The *quantum strategy framework* [9], which is also known as the *quantum combs framework* [2, 4], provides a useful framework for reasoning about networks of quantum channels. It may be used to model scenarios in which two or more entities, which we will call *players*, process and exchange quantum information over the course of multiple rounds of communication; and it is particularly useful when one wishes to consider an optimization over all possible behaviors of one player, for any given specification of the other player or players. Various developments, applications, and variants of the quantum strategy framework can be found in [1, 3, 5, 8, 10], for instance, and in a number of other sources.

In the discussion of the quantum strategy framework that follows, as well as in the subsequent sections of this paper, we assume that the reader is familiar with quantum information theory and semidefinite programming. References on this material include [11, 13, 15, 16] as well as [14], which we follow closely with respect to notation and terminology. In particular, we denote quantum registers by capital sans serif letters such as X, Y, and Z (sometimes with natural number subscripts), while the same letters (with matching subscripts) in a scripted font, such as $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ denote the complex Euclidean spaces (i.e., finite-dimensional complex Hilbert spaces) associated with the corresponding registers. The set $\mathrm{L}(\mathcal{X},\mathcal{Y})$ denotes the set of all linear operators from $\mathcal{X}$ to $\mathcal{Y}$; $\mathrm{L}(\mathcal{X})$

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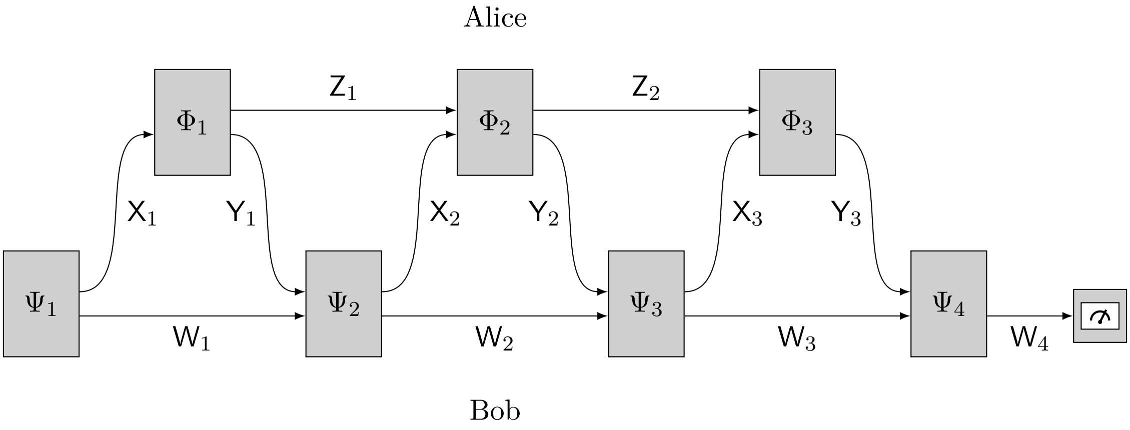

Figure 1: A six message interaction between Alice and Bob, after which Bob produces a measurement outcome.

is a shorthand for $\mathrm{L}(\mathcal{X},\mathcal{X})$; $\mathrm{Herm}(\mathcal{X})$, $\mathrm{Pos}(\mathcal{X})$, $\mathrm{D}(\mathcal{X})$, and $\mathrm{U}(\mathcal{X})$ denote the sets of all Hermitian operators, positive semidefinite operators, density operators, and unitary operators acting on $\mathcal{X}$; $\mathrm{C}(\mathcal{X},\mathcal{Y})$ denotes the set of all channels (i.e., completely positive and trace-preserving maps) mapping $\mathrm{L}(\mathcal{X})$ to $\mathrm{L}(\mathcal{Y})$; and $\mathrm{C}(\mathcal{X})$ is a shorthand for $\mathrm{C}(\mathcal{X},\mathcal{X})$. The adjoint of an operator A is denoted A^*, the entry-wise complex conjugate is denoted $\overline{A}$, and the transpose is denoted A^{T}. A similar notation is used for the adjoint and transpose of a channel Φ (the meaning of which, in the case of the transpose, will be clarified later). The (Hilbert-Schmidt) inner-product is defined as $\langle A, B\rangle = \mathrm{Tr}(A^*B)$ for all operators $A, B \in \mathrm{L}(\mathcal{X})$. Some additional notation will be introduced as it is used.

An example of a six-message interaction

To explain the aspects of the quantum strategy framework that are relevant to this paper, we will begin by discussing an example of an interaction structure involving six messages exchanged between two players, Alice and Bob. We have chosen to describe a six-message interaction because it is simple and concrete, but nevertheless clearly suggests the underlying structure of an interaction having any finite number of message exchanges. Our main result holds in the general case, which will be considered later, where an arbitrary finite number of message exchanges may take place.

Figure 1 illustrates an interaction between Alice and Bob. In this figure, time proceeds from left to right, and the arrows represent registers either being sent from one player to the other (as is the case for the registers X_1, Y_1, X_2, Y_2, X_3, and Y_3), or momentarily stored by one of the two players (as is the case for Z_1 and Z_2, stored by Alice, and W_1, W_2, W_3, W_4, stored by Bob). Alice's actions are represented by the channels Φ_1, Φ_2, and Φ_3, and Bob's actions are represented by the channels Ψ_1, Ψ_2, Ψ_3, and Ψ_4, as well as a final measurement, which is not given a name in the figure.

Suppose that Bob's specification has been fixed, including his choices for the channels Ψ_1, Ψ_2, Ψ_3, and Ψ_4, as well as his final measurement, and suppose further that one of Bob's possible measurement outcomes is to be viewed as desirable to Alice. It is then natural to consider an optimization over Alice's possible actions, maximizing the probability that Bob's measurement produces the outcome Alice desires. The quantum strategy framework reveals that this optimization problem can be expressed as a semidefinite program, in the manner that will now be described.

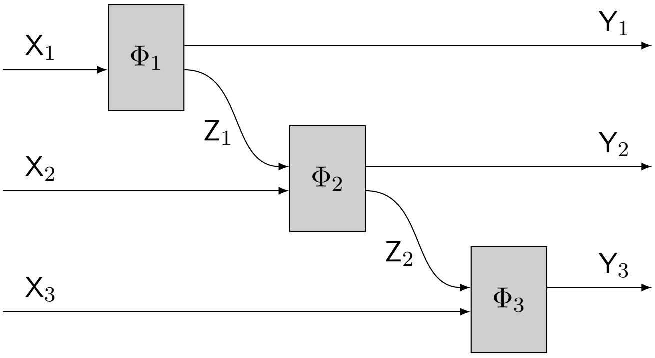

Figure 2: The channel Ξ_3 that describes Alice's actions in the interaction illustrated in Figure 1.

First, a single channel Ξ_3 that transforms $(\mathsf{X}_1, \mathsf{X}_2, \mathsf{X}_3)$ to $(\mathsf{Y}_1, \mathsf{Y}_2, \mathsf{Y}_3)$ is associated with any given choice for Alice's actions. That is, the channel Ξ_3 takes the form

$$\Xi_3 \in \mathrm{C}(\mathcal{X}_1 \otimes \mathcal{X}_2 \otimes \mathcal{X}_3, \mathcal{Y}_1 \otimes \mathcal{Y}_2 \otimes \mathcal{Y}_3), \tag{1}$$

and for a particular selection of Φ_1, Φ_2, and Φ_3 may be expressed as

$$\Xi_3 = \bigl(\mathbb{1}_{\mathrm{L}(\mathcal{Y}_1 \otimes \mathcal{Y}_2)} \otimes \Phi_3\bigr)\bigl(\mathbb{1}_{\mathrm{L}(\mathcal{Y}_1)} \otimes \Phi_2 \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X}_3)}\bigr)\bigl(\Phi_1 \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X}_2 \otimes \mathcal{X}_3)}\bigr). \tag{2}$$

Formally speaking, this composition requires that we view Φ_1, Φ_2, and Φ_3 as channels of the form $\Phi_1 \in \mathrm{C}(\mathcal{X}_1, \mathcal{Y}_1 \otimes \mathcal{Z}_1)$, $\Phi_2 \in \mathrm{C}(\mathcal{Z}_1 \otimes \mathcal{X}_2, \mathcal{Y}_2 \otimes \mathcal{Z}_2)$, and $\Phi_3 \in \mathrm{C}(\mathcal{Z}_2 \otimes \mathcal{X}_3, \mathcal{Y}_3)$, as opposed to the forms $\Phi_1 \in \mathrm{C}(\mathcal{X}_1, \mathcal{Z}_1 \otimes \mathcal{Y}_1)$, $\Phi_2 \in \mathrm{C}(\mathcal{Z}_1 \otimes \mathcal{X}_2, \mathcal{Z}_2 \otimes \mathcal{Y}_2)$, and $\Phi_3 \in \mathrm{C}(\mathcal{Z}_2 \otimes \mathcal{X}_3, \mathcal{Y}_3)$ suggested by Figure 1, so that the ordering of the tensor factors of the various input and output spaces is consistent with the composition. Similar re-orderings of tensor factors should be assumed implicitly throughout this paper as needed. This understanding should not be a source of confusion because we always assign distinct names to distinct registers (and their associated spaces). Figure 2 illustrates the action of the channel Ξ_3, which in words may be described as the channel obtained if all three of the registers $(\mathsf{X}_1, \mathsf{X}_2, \mathsf{X}_3)$ are provided initially, and then Alice's actions are composed in the natural way to produce $(\mathsf{Y}_1, \mathsf{Y}_2, \mathsf{Y}_3)$ as output registers.

It may appear that by considering the channel Ξ_3, one is ignoring the possibility that Bob's actions could, for instance, allow the contents of Y_1 or Y_2 to influence what is input into X_2 or X_3. Despite this appearance, the influence that Alice's actions have from the viewpoint of Bob, including the probability for each of his measurement outcomes to appear, is uniquely determined by the channel Ξ_3.

Naturally, not all channels of the form (1) will arise from a composition of channels Φ_1, Φ_2, and Φ_3 as in (2); the fact that Φ_1 is effectively performed first, Φ_2 is performed second, and Φ_3 is performed third imposes constraints on the channels Ξ_3 that can be obtained. In particular, consider the channel that results when Ξ_3 is performed and then the partial trace is performed on $\mathcal{Y}_3$. As Φ_3 is a channel, discarding its output is equivalent to discarding its inputs, from which it follows that

$$\mathrm{Tr}_{\mathcal{Y}_3} \circ \Xi_3 = \Xi_2 \circ \mathrm{Tr}_{\mathcal{X}_3}, \tag{3}$$

where the circles represent channel compositions and $\Xi_2 \in \mathrm{C}(\mathcal{X}_1 \otimes \mathcal{X}_2, \mathcal{Y}_1 \otimes \mathcal{Y}_2)$ is the channel defined as

$$\Xi_2 = \bigl(\mathbb{1}_{\mathrm{L}(\mathcal{Y}_1)} \otimes (\mathrm{Tr}_{\mathcal{Z}_2} \circ \Phi_2)\bigr)\bigl(\Phi_1 \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X}_2)}\bigr). \tag{4}$$

That is, Ξ_2 is the channel obtained from Φ_1 and Φ_2, followed by the partial trace over $\mathcal{Z}_2$, by a similar process to the one used to obtain Ξ_3. By similar reasoning, one finds that

$$\operatorname{Tr}_{\mathcal{Y}_2} \circ \Xi_2 = \Xi_1 \circ \operatorname{Tr}_{\mathcal{X}_2}, \tag{5}$$

where $\Xi_1 \in \mathrm{C}(\mathcal{X}_1, \mathcal{Y}_1)$ is the channel given by $\Xi_1 = \operatorname{Tr}_{\mathcal{Z}_1} \circ \Phi_1$.

Somewhat remarkably, this is not only a necessary condition on the channel Ξ_3, but also a sufficient one, for it to be obtained from a composition of channels Φ_1, Φ_2, and Φ_3 as described above. That is, given any channel

$$\Xi_3 \in \mathrm{C}(\mathcal{X}_1 \otimes \mathcal{X}_2 \otimes \mathcal{X}_3, \mathcal{Y}_1 \otimes \mathcal{Y}_2 \otimes \mathcal{Y}_3) \tag{6}$$

satisfying (3) and (5), for some choice of channels

$$\begin{gathered} \Xi_2 \in \mathrm{C}(\mathcal{X}_1 \otimes \mathcal{X}_2, \mathcal{Y}_1 \otimes \mathcal{Y}_2), \\ \Xi_1 \in \mathrm{C}(\mathcal{X}_1, \mathcal{Y}_1), \end{gathered} \tag{7}$$

there must exist channels

$$\begin{gathered} \Phi_1 \in \mathrm{C}(\mathcal{X}_1, \mathcal{Y}_1 \otimes \mathcal{Z}_1), \\ \Phi_2 \in \mathrm{C}(\mathcal{Z}_1 \otimes \mathcal{X}_2, \mathcal{Y}_2 \otimes \mathcal{Z}_2), \\ \Phi_3 \in \mathrm{C}(\mathcal{Z}_2 \otimes \mathcal{X}_3, \mathcal{Y}_3), \end{gathered} \tag{8}$$

for spaces $\mathcal{Z}_1$ and $\mathcal{Z}_2$ having sufficiently large dimension, so that (2) holds. This fact is proved in [2, 4, 9], and we note that a key idea through which this equivalence is proved may be found in [7].

The next step toward an expression of the optimization problem suggested above as a semidefinite program makes use of the Choi representation of channels. The Choi representation of the channel Ξ_3 takes the form

$$J(\Xi_3) \in \mathrm{Pos}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 \otimes \mathcal{Y}_3 \otimes \mathcal{X}_1 \otimes \mathcal{X}_2 \otimes \mathcal{X}_3), \tag{9}$$

as the complete positivity of Φ_1, Φ_2, and Φ_3 implies that Ξ_3 is also completely positive, and therefore $J(\Xi_3)$ is positive semidefinite. The constraints on the channel Ξ_3 described previously correspond (very conveniently) to linear constraints; one has that (3) and (5) hold, for some choice of channels Ξ_2 and Ξ_1, if and only if the Choi representation $X_3 = J(\Xi_3)$ of Ξ_3 satisfies

$$\begin{gathered} \operatorname{Tr}_{\mathcal{Y}_3}(X_3) = X_2 \otimes \mathbb{1}_{\mathcal{X}_3}, \\ \operatorname{Tr}_{\mathcal{Y}_2}(X_2) = X_1 \otimes \mathbb{1}_{\mathcal{X}_2}, \\ \operatorname{Tr}_{\mathcal{Y}_1}(X_1) = \mathbb{1}_{\mathcal{X}_1}, \end{gathered} \tag{10}$$

for some choice of operators

$$\begin{gathered} X_2 \in \mathrm{Pos}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 \otimes \mathcal{X}_1 \otimes \mathcal{X}_2), \\ X_1 \in \mathrm{Pos}(\mathcal{Y}_1 \otimes \mathcal{X}_1). \end{gathered} \tag{11}$$

As is to be expected, the operators X_2 and X_1 correspond to the Choi representations $X_2 = J(\Xi_2)$ and $X_1 = J(\Xi_1)$.

Finally, the probability that Bob's measurement produces any one fixed outcome is a linear function of the channel Ξ_3, and is therefore a linear function of the Choi representation $X_3 = J(\Xi_3)$. Although this process is not relevant to the main result of this paper, we note that it is possible to obtain an explicit description of this linear function given a specification of Bob's actions, including his final measurement. In somewhat vague terms, the linear function describing Bob's probability to produce a particular measurement outcome is given by $\langle P, X_3 \rangle$, where

$$P \in \mathrm{Pos}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 \otimes \mathcal{Y}_3 \otimes \mathcal{X}_1 \otimes \mathcal{X}_2 \otimes \mathcal{X}_3) \tag{12}$$

is an operator that is obtained from Ψ_1, Ψ_2, Ψ_3, Ψ_4, and the measurement operator corresponding to the outcome being considered by a process very similar to the one through which X_3 is obtained from Φ_1, Φ_2, and Φ_3. Once again, the reader is referred to [2, 4, 9] for further details.

More generally, an arbitrary real-valued linear function of the operator X_3 may be expressed as $\langle H, X_3 \rangle$ for some choice of a Hermitian operator

$$H \in \mathrm{Herm}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 \otimes \mathcal{Y}_3 \otimes \mathcal{X}_1 \otimes \mathcal{X}_2 \otimes \mathcal{X}_3), \tag{13}$$

which need not represent the probability with which a particular measurement outcome is obtained for channels $\Psi_1, \ldots, \Psi_4$ followed by a measurement. Such a function could, for instance, represent an expected payoff for Alice's actions, under the assumption that a real-valued payoff is associated with each of Bob's measurement outcomes.

General semidefinite programming formulation

As mentioned previously, the six-message example just described generalizes to any finite number of message exchanges. If the number of message exchanges is equal to n, the input registers to Alice (the player whose actions are being optimized) are $\mathsf{X}_1, \ldots, \mathsf{X}_n$, and the output registers of Alice are $\mathsf{Y}_1, \ldots, \mathsf{Y}_n$, then the possible strategies for Alice are represented by channels of the form

$$\Xi_n \in \mathrm{C}(\mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n, \mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_n) \tag{14}$$

that obey constraints that generalize (3) and (5). Specifically, there must exist channels

$$\begin{gathered}\Xi_{n-1} \in \mathrm{C}(\mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_{n-1}, \mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_{n-1})\\ \vdots \\ \Xi_1 \in \mathrm{C}(\mathcal{X}_1, \mathcal{Y}_1)\end{gathered} \tag{15}$$

such that

$$\mathrm{Tr}_{\mathcal{Y}_k} \circ \Xi_k = \Xi_{k-1} \circ \mathrm{Tr}_{\mathcal{X}_k} \tag{16}$$

for all $k \in \{2, \ldots, n\}$. For the maximization of a real-valued linear function over all strategies for Alice, represented by a Hermitian operator

$$H \in \mathrm{Herm}(\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_n \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n), \tag{17}$$

one obtains the semidefinite program described in Figure 3. The primal problem corresponds to an optimization over all Choi representations of the channels $\Xi_1, \ldots, \Xi_n$. This semidefinite programming formulation is implicit in [9], and first appeared explicitly in [8]. It also appears in [1], where it was used to define a generalized notion of min-entropy for quantum networks.

It may be noted that the general problem just formulated concerns interactions involving an even number of register exchanges, where Alice (the player whose actions are being optimized) always receives the first transmission, represented by X_1, and sends the last transmission, represented by Y_n. However, one is free to take either or both of the registers X_1 and Y_n to be trivial registers, so that correspondingly $\mathcal{X}_1 = \mathbb{C}$ and/or $\mathcal{Y}_n = \mathbb{C}$. This is tantamount to allowing either an odd number of register exchanges or an even number in the situation that Alice sends the first (nontrivial) register and receives the last.

Primal problem

$$\begin{aligned}
\text{maximize:}\quad & \langle H, X_n \rangle \\
\text{subject to:}\quad & \operatorname{Tr}_{\mathcal{Y}_n}(X_n) = X_{n-1} \otimes \mathbb{1}_{\mathcal{X}_n}, \\
& \qquad \vdots \\
& \operatorname{Tr}_{\mathcal{Y}_2}(X_2) = X_1 \otimes \mathbb{1}_{\mathcal{X}_2}, \\
& \operatorname{Tr}_{\mathcal{Y}_1}(X_1) = \mathbb{1}_{\mathcal{X}_1}, \\
& X_n \in \operatorname{Pos}(\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_n \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n), \\
& \qquad \vdots \\
& X_2 \in \operatorname{Pos}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 \otimes \mathcal{X}_1 \otimes \mathcal{X}_2), \\
& X_1 \in \operatorname{Pos}(\mathcal{Y}_1 \otimes \mathcal{X}_1).
\end{aligned}$$

Dual problem

$$\begin{aligned}
\text{minimize:}\quad & \operatorname{Tr}(Y_1) \\
\text{subject to:}\quad & Y_n \otimes \mathbb{1}_{\mathcal{Y}_n} \geq H, \\
& Y_{n-1} \otimes \mathbb{1}_{\mathcal{Y}_{n-1}} \geq \operatorname{Tr}_{\mathcal{X}_n}(Y_n), \\
& \qquad \vdots \\
& Y_1 \otimes \mathbb{1}_{\mathcal{Y}_1} \geq \operatorname{Tr}_{\mathcal{X}_2}(Y_2), \\
& Y_n \in \operatorname{Herm}(\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_{n-1} \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n), \\
& Y_{n-1} \in \operatorname{Herm}(\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_{n-2} \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_{n-1}), \\
& \qquad \vdots \\
& Y_1 \in \operatorname{Herm}(\mathcal{X}_1).
\end{aligned}$$

Figure 3: The semidefinite program representing a maximization of a linear function of an n-turn strategy.

2 Statement and proof of the main result

The main result of the current paper concerns the optimization problem described in the previous section, as represented by the semidefinite program in Figure 3, in the case that $H = uu^*$ is a rank one positive semidefinite operator. The result to be described does not hold in general when H does not take this form.

In order to explain the main result in precise terms, it will be helpful to introduce some notation. Suppose that a positive integer n along with spaces $\mathcal{X}_1, \ldots, \mathcal{X}_n$ and $\mathcal{Y}_1, \ldots, \mathcal{Y}_n$ have been fixed. For each $k \in \{1, \ldots, n\}$, let

$$\mathcal{S}_k(\mathcal{X}_1, \ldots, \mathcal{X}_k; \mathcal{Y}_1, \ldots, \mathcal{Y}_k) \subset \operatorname{Pos}(\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_k \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k) \tag{18}$$

denote the primal-feasible choices for the operator X_k in the semidefinite program specified in Figure 3. That is, we define

$$\mathcal{S}_1(\mathcal{X}_1; \mathcal{Y}_1) = \{X_1 \in \operatorname{Pos}(\mathcal{Y}_1 \otimes \mathcal{X}_1) \,:\, \operatorname{Tr}_{\mathcal{Y}_1}(X_1) = \mathbb{1}_{\mathcal{X}_1}\} \tag{19}$$

(which is the set of all Choi operators of channels of the form $\Xi_1 \in \mathrm{C}(\mathcal{X}_1, \mathcal{Y}_1)$), and

$$\begin{aligned}&\mathcal{S}_k(\mathcal{X}_1, \ldots, \mathcal{X}_k; \mathcal{Y}_1, \ldots, \mathcal{Y}_k)\\&\quad = \bigl\{X_k \in \mathrm{Pos}(\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_k \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k) \,:\, \mathrm{Tr}_{\mathcal{Y}_k}(X_k) = X_{k-1} \otimes \mathbb{1}_{\mathcal{X}_k}\\&\qquad\qquad \text{for some } X_{k-1} \in \mathcal{S}_{k-1}(\mathcal{X}_1, \ldots, \mathcal{X}_{k-1}; \mathcal{Y}_1, \ldots, \mathcal{Y}_{k-1})\bigr\}\end{aligned} \tag{20}$$

for $k \in \{2, \ldots, n\}$. The primal form of the semidefinite program described in Figure 3 can therefore be expressed succinctly as

$$\begin{aligned}\text{maximize:}\quad & \langle H, X\rangle\\ \text{subject to:}\quad & X \in \mathcal{S}_n(\mathcal{X}_1, \ldots, \mathcal{X}_n; \mathcal{Y}_1, \ldots, \mathcal{Y}_n).\end{aligned} \tag{21}$$

We will refer to operators in the sets defined above as *strategy operators*, as they represent n-turn strategies with respect to the quantum strategy framework.

Let us also define an isometry

$$W \in \mathrm{U}(\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_n \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n, \mathcal{X}_n \otimes \cdots \otimes \mathcal{X}_1 \otimes \mathcal{Y}_n \otimes \cdots \otimes \mathcal{Y}_1) \tag{22}$$

by the action

$$W(y_1 \otimes \cdots \otimes y_n \otimes x_1 \otimes \cdots \otimes x_n) = x_n \otimes \cdots \otimes x_1 \otimes y_n \otimes \cdots \otimes y_1 \tag{23}$$

for all vectors $x_1 \in \mathcal{X}_1, \ldots, x_n \in \mathcal{X}_n$ and $y_1 \in \mathcal{Y}_1, \ldots, y_n \in \mathcal{Y}_n$. In words, W simply reverses the order of the tensor factors of the space $\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_n \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n$, yielding a vector in $\mathcal{X}_n \otimes \cdots \otimes \mathcal{X}_1 \otimes \mathcal{Y}_n \otimes \cdots \otimes \mathcal{Y}_1$ that, aside from this re-ordering of tensor factors, is the same as its input vector.

Statement of the main result

With the notation just introduced in hand, the main theorem may now be stated.

Theorem 1. *Let $\mathcal{X}_1, \ldots, \mathcal{X}_n$ and $\mathcal{Y}_1, \ldots, \mathcal{Y}_n$ be complex Euclidean spaces, for n a positive integer, let*

$$u \in \mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_n \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n \tag{24}$$

be a vector, and let

$$X \in \mathcal{S}_n(\mathcal{X}_1, \ldots, \mathcal{X}_n; \mathcal{Y}_1, \ldots, \mathcal{Y}_n) \tag{25}$$

be a strategy operator. There exists a strategy operator

$$Y \in \mathcal{S}_n(\mathcal{Y}_n, \ldots, \mathcal{Y}_1; \mathcal{X}_n, \ldots, \mathcal{X}_1) \tag{26}$$

such that

$$\langle W uu^* W^*, Y\rangle \geq \langle uu^*, X\rangle. \tag{27}$$

If it is the case that $\dim(\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_n) \leq \dim(\mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n)$, then the operator Y may be chosen so that equality holds in (27).

Corollary 2. *Let $\mathcal{X}_1, \ldots, \mathcal{X}_n$ and $\mathcal{Y}_1, \ldots, \mathcal{Y}_n$ be complex Euclidean spaces, for n a positive integer, and let*

$$u \in \mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_n \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n \tag{28}$$

be a vector. The semidefinite optimization problems

$$\begin{aligned}\textit{maximize:}\quad & \langle uu^*, X\rangle\\ \textit{subject to:}\quad & X \in \mathcal{S}_n(\mathcal{X}_1, \ldots, \mathcal{X}_n; \mathcal{Y}_1, \ldots, \mathcal{Y}_n)\end{aligned} \tag{29}$$

and

$$\begin{aligned}&\textit{maximize:} && \langle W u u^* W^*, Y\rangle\\ &\textit{subject to:} && Y \in \mathcal{S}_n(\mathcal{Y}_n,\ldots,\mathcal{Y}_1;\mathcal{X}_n,\ldots,\mathcal{X}_1)\end{aligned}\tag{30}$$

have the same optimum value.

Remark. Using the notation introduced in [1], which defines a quantum network generalization of conditional min-entropy, the equivalence expressed by Corollary 2 may alternatively be written

$$\begin{aligned}&\mathrm{H}_{\min}(\mathsf{Y}_n \mid \mathsf{X}_1,\mathsf{Y}_1,\ldots,\mathsf{X}_{n-1},\mathsf{Y}_{n-1},\mathsf{X}_n)_{uu^*}\\ &\qquad = \mathrm{H}_{\min}(\mathsf{X}_1 \mid \mathsf{Y}_n,\mathsf{X}_n,\ldots,\mathsf{Y}_2,\mathsf{X}_2,\mathsf{Y}_1)_{uu^*}\end{aligned}\tag{31}$$

for every vector $u \in \mathcal{X}_1 \otimes \mathcal{Y}_1 \otimes \cdots \otimes \mathcal{X}_n \otimes \mathcal{Y}_n$.

Interpretations of the main theorem

Theorem 1 establishes a *time-reversal property* of rank-one strategy functions. Intuitively speaking, the linear function

$$Y \mapsto \langle W u u^* W^*, Y\rangle \tag{32}$$

defined on $\mathcal{S}_n(\mathcal{Y}_n,\ldots,\mathcal{Y}_1;\mathcal{X}_n,\ldots,\mathcal{X}_1)$ represents the *time-reversal* of the linear function

$$X \mapsto \langle u u^*, X\rangle \tag{33}$$

defined on $\mathcal{S}_n(\mathcal{X}_1,\ldots,\mathcal{X}_n;\mathcal{Y}_1,\ldots,\mathcal{Y}_n)$, in the sense that the two functions differ only in the reversal of the ordering of the register exchanges: X_1, Y_1, …, X_n, Y_n for the function corresponding to uu^* and Y_n, X_n, …, Y_1, X_1 for the function corresponding to Wuu^*W^*.

For a given choice of $X \in \mathcal{S}_n(\mathcal{X}_1,\ldots,\mathcal{X}_n;\mathcal{Y}_1,\ldots,\mathcal{Y}_n)$, it is generally not the case that $W^*XW \in \mathcal{S}_n(\mathcal{Y}_n,\ldots,\mathcal{Y}_1;\mathcal{X}_n,\ldots,\mathcal{X}_1)$. It may not even be the case that W^*XW is the Choi representation of a channel, and in the case that W^*XW is the Choi representation of a channel, it will generally not be the case that this channel obeys the constraints necessary for it to be a valid strategy operator. When combined with the observation that $\mathcal{S}_n(\mathcal{X}_1,\ldots,\mathcal{X}_n;\mathcal{Y}_1,\ldots,\mathcal{Y}_n)$ and $\mathcal{S}_n(\mathcal{Y}_n,\ldots,\mathcal{Y}_1;\mathcal{X}_n,\ldots,\mathcal{X}_1)$ are compact and convex sets, this fact implies that the main theorem cannot possibly hold for all Hermitian operators H by the separating hyperplane theorem. For small values of n and for spaces having small dimensions, simple examples of operators H for which the main theorem fails may also easily be obtained through random selections.

In Section 4 we discuss another interpretation of Theorem 1, which concerns multiple round entanglement manipulation.

Proof of Theorem 1

We will now prove Theorem 1. The first step is to express the strategy represented by X as a sequence of channels corresponding to invertible isometries (i.e., unitary operators for which the input and output spaces have different names but necessarily the same dimension), assuming an auxiliary input space initialized to a pure state is made available.

Through the repeated application of the Stinespring dilation theorem, together with the result of [2, 4, 9] establishing that $X = J(\Xi_n)$ is the Choi representation of a channel Ξ_n arising from a valid n-turn strategy, one finds that there must exist complex Euclidean spaces $\mathcal{Z}_0,\ldots,\mathcal{Z}_n$ satisfying $\dim(\mathcal{Z}_{k-1}\otimes\mathcal{X}_k) = \dim(\mathcal{Z}_k\otimes\mathcal{Y}_k)$ for all $k \in \{1,\ldots,n\}$, a unit vector $v \in \mathcal{Z}_0$, and invertible isometries $U_1,\ldots,U_n$ of the form

$$U_k \in \mathrm{U}(\mathcal{Z}_{k-1}\otimes\mathcal{X}_k, \mathcal{Y}_k\otimes\mathcal{Z}_k) \tag{34}$$

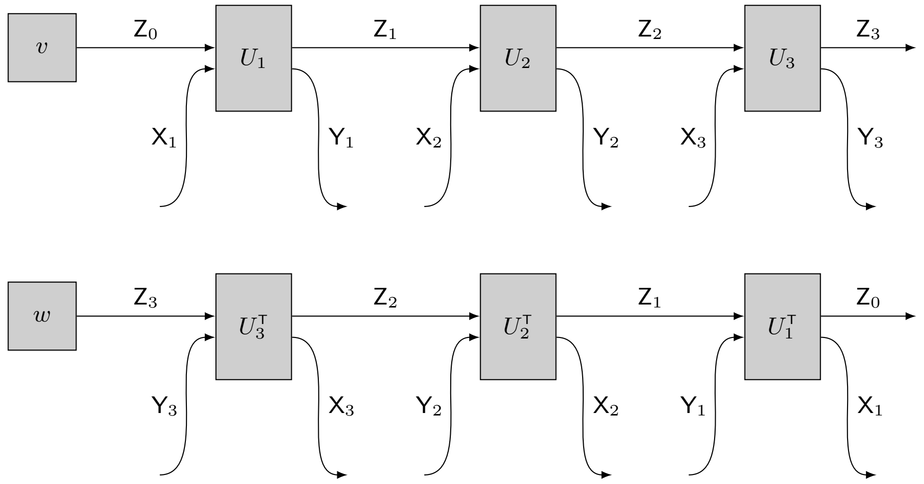

Figure 4: An arbitrary strategy may be implemented by initializing a register Z_0 to a pure state v, followed by the application of an invertible isometric channel on each turn, and finally by discarding the last memory register Z_n (which is Z_3 in the picture). The time-reversed strategy whose existence is implied by the main theorem is obtained by setting the register Z_n (Z_3 in the picture) to an appropriate choice of a pure state w, followed by the application of invertible isometric channels obtained by transposing the original isometries, and finally by discarding the memory register Z_0.

such that

$$\Xi_n(Z) = \operatorname{Tr}_{\mathcal{Z}_n}\bigl(U(vv^* \otimes Z)U^*\bigr) \tag{35}$$

for all $Z \in \mathrm{L}(\mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n)$, where

$$\begin{aligned} U &= (\mathbb{1}_{\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_{n-1}} \otimes U_n) \cdots (U_1 \otimes \mathbb{1}_{\mathcal{X}_2 \otimes \cdots \otimes \mathcal{X}_n}) \\ &\in \mathrm{U}(\mathcal{Z}_0 \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n, \mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_n \otimes \mathcal{Z}_n). \end{aligned} \tag{36}$$

In words, the strategy represented by the operator X is implemented by first initializing a register Z_0 to the pure state v, then applying the invertible isometric channels corresponding to $U_1, \ldots, U_n$, and finally discarding Z_n after the interaction has finished. (The top picture in Figure 4 illustrates this for the case $n = 3$.)

The vector u may be expressed as

$$u = \sum_{\substack{a_1,\ldots,a_n \\ b_1,\ldots,b_n}} u(b_1,\ldots,b_n,a_1,\ldots,a_n)|b_1\rangle \cdots |b_n\rangle|a_1\rangle \cdots |a_n\rangle, \tag{37}$$

where the sum is over all standard basis states $|a_1\rangle, \ldots, |a_n\rangle$ of $\mathcal{X}_1, \ldots, \mathcal{X}_n$ and $|b_1\rangle, \ldots, |b_n\rangle$ of $\mathcal{Y}_1, \ldots, \mathcal{Y}_n$, respectively. Based on this expression, define an operator $A \in \mathrm{L}(\mathcal{Z}_0, \mathcal{Z}_n)$ as

$$\begin{aligned} A = \sum_{\substack{a_1,\ldots,a_n \\ b_1,\ldots,b_n}} u(b_1,\ldots,b_n,a_1,\ldots,a_n)(\langle b_n| \otimes \mathbb{1}_{\mathcal{Z}_n})U_n(\mathbb{1}_{\mathcal{Z}_{n-1}} \otimes |a_n\rangle) \\ \cdots (\langle b_1| \otimes \mathbb{1}_{\mathcal{Z}_1})U_1(\mathbb{1}_{\mathcal{Z}_0} \otimes |a_1\rangle). \end{aligned} \tag{38}$$

By considering the action of the strategy represented by v and $U_1, \ldots, U_n$, then performing the required operator-vector multiplications required to evaluate the expression $\langle uu^*, X\rangle$ when $X = J(\Xi_n)$ for Ξ_n given by (35), one concludes that

$$\langle uu^*, X\rangle = \|Av\|^2 = \langle vv^*, A^*A\rangle. \tag{39}$$

Next we turn to the reversed interaction. To obtain a strategy operator Y satisfying the requirements of the theorem, we consider the strategy obtained by initializing the register

Z_n to a particular choice of a pure state w, which will be selected later, then applying in sequence the invertible isometric channels corresponding to the operators $U_n^{\mathsf{T}}, \ldots, U_1^{\mathsf{T}}$. (The bottom picture in Figure 4 illustrates this for the case $n = 3$.) That is, for

$$\begin{aligned} V &= (\mathbb{1}_{\mathcal{X}_n \otimes \cdots \otimes \mathcal{X}_2} \otimes U_1^{\mathsf{T}}) \cdots (U_n^{\mathsf{T}} \otimes \mathbb{1}_{\mathcal{Y}_{n-1} \otimes \cdots \otimes \mathcal{Y}_1}) \\ &\in \mathrm{U}(\mathcal{Z}_n \otimes \mathcal{Y}_n \otimes \cdots \otimes \mathcal{Y}_1, \mathcal{Z}_0 \otimes \mathcal{X}_n \otimes \cdots \otimes \mathcal{X}_1), \end{aligned} \tag{40}$$

we consider the channel $\Theta_n \in \mathrm{C}(\mathcal{Y}_n \otimes \cdots \otimes \mathcal{Y}_1, \mathcal{X}_n \otimes \cdots \otimes \mathcal{X}_1)$ defined as

$$\Theta_n(Z) = \operatorname{Tr}_{\mathcal{Z}_0}\big(V(ww^* \otimes Z)V^*\big) \tag{41}$$

for all $Z \in \mathrm{L}(\mathcal{Y}_n \otimes \cdots \otimes \mathcal{Y}_1)$. It is evident from the specification of this channel, irrespective of the choice of the pure state w, that $Y = J(\Theta_n) \in \mathcal{S}_n(\mathcal{Y}_n, \ldots, \mathcal{Y}_1; \mathcal{X}_n, \ldots, \mathcal{X}_1)$. By considering the action of this strategy, a similar calculation to the one above reveals that

$$\langle Wuu^*W^*, Y \rangle = \|A^{\mathsf{T}} w\|^2 = \langle ww^*, \overline{A}A^{\mathsf{T}} \rangle. \tag{42}$$

The nonzero eigenvalues of A^*A and $\overline{A}A^{\mathsf{T}}$ are equal, and therefore by choosing w to be an eigenvector corresponding to the largest eigenvalue of $\overline{A}A^{\mathsf{T}}$ one obtains

$$\langle Wuu^*W^*, Y \rangle = \langle ww^*, \overline{A}A^{\mathsf{T}} \rangle \geq \langle vv^*, A^*A \rangle = \langle uu^*, X \rangle. \tag{43}$$

If it holds that $\dim(\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_n) \leq \dim(\mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n)$, then $\dim(\mathcal{Z}_0) \leq \dim(\mathcal{Z}_n)$, which implies that the inequality in (43) may be taken as an equality for an appropriate choice of a pure state w. This completes the proof.

3 Application to min- and max-entropy

In this section we connect the main result proved in the previous section to the conditional min- and max-entropy functions. These function, which were first introduced in [6], may be defined as follows. First, one defines the max- and min-relative entropy of P with respect to Q, for positive semidefinite operators P and Q (acting on the same space), as follows:

$$\mathrm{D}_{\max}(P \,\|\, Q) = \log\big(\min\{\lambda \geq 0 \,:\, P \leq \lambda Q\}\big), \tag{44}$$

$$\mathrm{D}_{\min}(P \,\|\, Q) = -\log\big(\mathrm{F}(P, Q)^2\big). \tag{45}$$

Then, with respect to a given state $\rho \in \mathrm{D}(\mathcal{X} \otimes \mathcal{Y})$ of a pair of registers (X, Y), one defines

$$\mathrm{H}_{\min}(\mathsf{X}|\mathsf{Y}) = -\inf_{\sigma \in \mathrm{D}(\mathcal{Y})} \mathrm{D}_{\max}(\rho \,\|\, \mathbb{1}_{\mathcal{X}} \otimes \sigma), \tag{46}$$

$$\mathrm{H}_{\max}(\mathsf{X}|\mathsf{Y}) = -\inf_{\sigma \in \mathrm{D}(\mathcal{Y})} \mathrm{D}_{\min}(\rho \,\|\, \mathbb{1}_{\mathcal{X}} \otimes \sigma). \tag{47}$$

It is known that these two quantities are related in the following way: with respect to any pure state uu^* of a triple of registers $(\mathsf{X}, \mathsf{Y}, \mathsf{Z})$, one has that

$$\mathrm{H}_{\min}(\mathsf{X}|\mathsf{Y}) = -\,\mathrm{H}_{\max}(\mathsf{X}|\mathsf{Z}). \tag{48}$$

(Indeed, in [12] the conditional max-relative entropy of a state of (X, Z) is *defined* by the equation (48), which does not depend on which purification of this state is chosen, and is then proved to agree with the definition stated previously.)

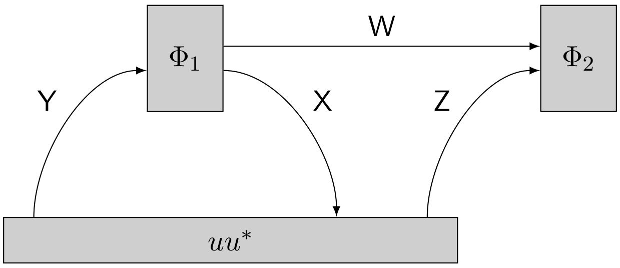

Figure 5: The optimization problem (49) corresponds to a maximization of the linear functions defined by uu^* over all strategies given by channels Φ_1 and Φ_2, for an arbitrary choice of a register W.

Consider any unit vector $u \in \mathcal{X} \otimes \mathcal{Y} \otimes \mathcal{Z}$, which defines a pure state uu^* of a triple of registers $(\mathsf{X}, \mathsf{Y}, \mathsf{Z})$. We will consider two optimization problems defined by u, the first of which is as follows:

$$\begin{aligned} \text{maximize:} \quad & \langle uu^*, X \rangle \\ \text{subject to:} \quad & X \in \mathcal{S}_2(\mathcal{Y}, \mathcal{Z}; \mathcal{X}, \mathbb{C}). \end{aligned} \tag{49}$$

This optimization problem is illustrated in Figure 5. In this case, the channel Φ_2 takes registers Z and W as input and outputs nothing (which is equivalent to outputting the unique state $1 \in \mathrm{D}(\mathbb{C})$ of a one-dimensional system). That is, Φ_2 must be the trace mapping. One may therefore simplify this problem, obtaining the following semidefinite program:

Primal problem

$$\begin{aligned} \text{maximize:} \quad & \langle \mathrm{Tr}_{\mathcal{Z}}(uu^*), X \rangle \\ \text{subject to:} \quad & \mathrm{Tr}_{\mathcal{X}}(X) = \mathbb{1}_{\mathcal{Y}}, \\ & X \in \mathrm{Pos}(\mathcal{X} \otimes \mathcal{Y}). \end{aligned}$$

Dual problem

$$\begin{aligned} \text{minimize:} \quad & \mathrm{Tr}(Y) \\ \text{subject to:} \quad & \mathbb{1}_{\mathcal{X}} \otimes Y \geq \mathrm{Tr}_{\mathcal{Z}}(uu^*), \\ & Y \in \mathrm{Herm}(\mathcal{X}). \end{aligned}$$

By examining the dual problem, one sees that the optimal value of this semidefinite program is

$$2^{-\mathrm{H}_{\min}(\mathsf{X}|\mathsf{Y})} \tag{50}$$

with respect to the state uu^* of $(\mathsf{X}, \mathsf{Y}, \mathsf{Z})$. König, Renner, and Schaffner [12] observed that the primal problem coincides with the value represented by the expression (50), which is consistent with the observation that strong duality always holds for this semidefinite program (which may be verified through Slater's theorem, for instance).

The second optimization problem we consider is the time-reversal of the first, and may be stated as follows:

$$\begin{aligned} \text{maximize:} \quad & \langle W uu^* W^*, Y \rangle \\ \text{subject to:} \quad & Y \in \mathcal{S}_2(\mathbb{C}, \mathcal{X}; \mathcal{Z}, \mathcal{Y}). \end{aligned} \tag{51}$$

Figure 6 illustrates the interaction corresponding to this optimization problem. The inclusion $X \in \mathcal{S}_2(\mathbb{C}, \mathcal{X}; \mathcal{Z}, \mathcal{Y})$, for a given operator $X \in \mathrm{Pos}(\mathcal{Z} \otimes \mathcal{Y} \otimes \mathcal{X})$, is equivalent to the condition that $\mathrm{Tr}_{\mathcal{Y}}(X) = \sigma \otimes \mathbb{1}_{\mathcal{X}}$ for some $\sigma \in \mathrm{D}(\mathcal{Z})$. After re-ordering tensor factors, we obtain the following semidefinite program:

Primal problem

$$\begin{aligned} \text{maximize:} \quad & \langle uu^*, X \rangle \\ \text{subject to:} \quad & \mathrm{Tr}_{\mathcal{Y}}(X) = \mathbb{1}_{\mathcal{X}} \otimes \sigma, \\ & X \in \mathrm{Pos}(\mathcal{X} \otimes \mathcal{Y} \otimes \mathcal{Z}), \\ & \sigma \in \mathrm{D}(\mathcal{Z}). \end{aligned}$$

Dual problem

$$\begin{aligned} \text{minimize:} \quad & \lambda \\ \text{subject to:} \quad & Y \otimes \mathbb{1}_{\mathcal{Y}} \geq uu^*, \\ & \lambda \mathbb{1}_{\mathcal{Z}} \geq \mathrm{Tr}_{\mathcal{X}}(Y), \\ & Y \in \mathrm{Herm}(\mathcal{X} \otimes \mathcal{Z}), \\ & \lambda \in \mathbb{R}. \end{aligned}$$

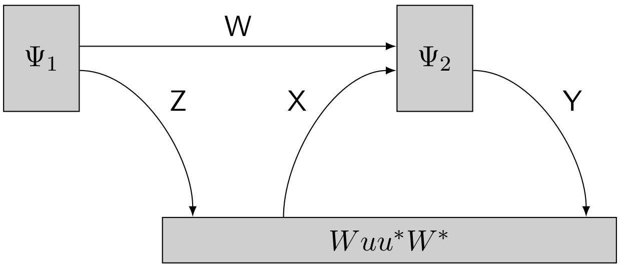

Figure 6: The optimization problem (51) corresponds to a maximization of the linear functions defined by Wuu^*W^* over all strategies given by channels Ψ_1 and Ψ_2, for an arbitrary choice of a register W.

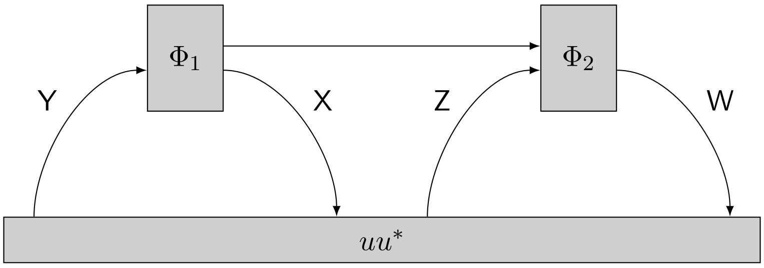

Figure 7: Maximizing the linear function defined by uu^* over all four-message strategies of the form depicted yields the left-hand side of (54). By reversing time, the right-hand side of that equation is obtained, and the equality of the two is implied by the main theorem.

An examination of the primal problem reveals (through Uhlmann's theorem) that the optimal value of this semidefinite program is

$$2^{\mathrm{H}_{\max}(\mathsf{X}|\mathsf{Z})}. \tag{52}$$

By our main theorem, it follows that the two optimization problems have the same optimal value, and therefore we obtain an alternative proof that with respect to every pure state of a triple or registers $(\mathsf{X}, \mathsf{Y}, \mathsf{Z})$ one has

$$\mathrm{H}_{\min}(\mathsf{X}|\mathsf{Y}) = -\,\mathrm{H}_{\max}(\mathsf{X}|\mathsf{Z}). \tag{53}$$

It is natural to ask if the connections among min-entropy, max-entropy, and optimization problems involving three-message strategies have interesting implications or generalizations for interactions involving four or more messages. As a partial answer to this question, we observe that when our main result is applied to the four-message interaction depicted in Figure 7, it reveals the identity

$$\max_{\Phi \in \mathrm{C}(\mathcal{Y},\mathcal{X})} \mathrm{F}\big(\mathrm{Tr}_{\mathcal{W}}(uu^*), J(\Phi) \otimes \mathbb{1}_{\mathcal{Z}}\big) = \max_{\Psi \in \mathrm{C}(\mathcal{W},\mathcal{Z})} \mathrm{F}\big(\mathrm{Tr}_{\mathcal{X}}(uu^*), \mathbb{1}_{\mathcal{Y}} \otimes J(\Psi)\big) \tag{54}$$

for all vectors $u \in \mathcal{X} \otimes \mathcal{Y} \otimes \mathcal{Z} \otimes \mathcal{W}$. This identity is appealing in its simplicity and symmetry, and by taking $\mathcal{W} = \mathbb{C}$ (or $\mathcal{Y} = \mathbb{C}$) a statement equivalent to (53) for all pure states of $(\mathsf{X}, \mathsf{Y}, \mathsf{Z})$ is obtained. We do not know, however, if the quantity represented by either side of the identity has any direct operational significance.

Other identities may be obtained through a similar methodology, although they become increasingly complex as the number of messages is increased.

4 Online pure state entanglement manipulation

The following three statements are equivalent for a given operator $X \in \mathrm{L}(\mathcal{Y} \otimes \mathcal{X})$:

1. $X \in \mathcal{S}_1(\mathcal{X}; \mathcal{Y})$. (Equivalently, $X \in \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{X})$ and $\mathrm{Tr}_{\mathcal{Y}}(X) = \mathbb{1}_{\mathcal{X}}$.)
2. $X = (\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X})})(\mathrm{vec}(\mathbb{1}_{\mathcal{X}})\,\mathrm{vec}(\mathbb{1}_{\mathcal{X}})^*)$ for some channel $\Phi \in \mathrm{C}(\mathcal{X}, \mathcal{Y})$.
3. $X = (\mathbb{1}_{\mathrm{L}(\mathcal{Y})} \otimes \Psi)(\mathrm{vec}(\mathbb{1}_{\mathcal{Y}})\,\mathrm{vec}(\mathbb{1}_{\mathcal{Y}})^*)$ for some completely positive and unital map $\Psi \in \mathrm{CP}(\mathcal{Y}, \mathcal{X})$.

(Here and throughout this section, vec refers to the vectorization mapping, which is the mapping obtained by extending the transformation $|a\rangle\langle b| \mapsto |a\rangle|b\rangle$ for standard basis states to arbitrary operators by linearity. In particular, $\mathrm{vec}(\mathbb{1}_{\mathcal{X}})$ is a non-normalized vector proportional to the canonical maximally entangled pure state corresponding to two identical copies of a system whose state space is $\mathcal{X}$.) The maps Φ and Ψ uniquely determine one another, and it is reasonable to view these maps as being related by transposition (with respect to the standard basis): $\Psi = \Phi^{\mathsf{T}}$ and $\Phi = \Psi^{\mathsf{T}}$. To obtain a Kraus representation for Ψ, for instance, one may simply take a Kraus representation of Φ and transpose each of the Kraus operators. (The transpose of an arbitrary map can be defined in a manner that is consistent with these statements, but it is sufficient for our needs to focus on channels and completely positive unital maps.)

A generalization of the equivalence mentioned above to the quantum strategy framework may also be verified. For an operator $X \in \mathrm{L}(\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_n \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n)$, these three statements are equivalent:

1. $X \in \mathcal{S}_n(\mathcal{X}_1, \ldots, \mathcal{X}_n; \mathcal{Y}_1, \ldots, \mathcal{Y}_n)$.
2. There exist complex Euclidean spaces $\mathcal{Z}_1, \ldots, \mathcal{Z}_{n-1}$ (and $\mathcal{Z}_0 = \mathbb{C}$ and $\mathcal{Z}_n = \mathbb{C}$), along with channels $\Phi_1, \ldots, \Phi_n$ having the form
$$\Phi_k \in \mathrm{C}(\mathcal{Z}_{k-1} \otimes \mathcal{X}_k, \mathcal{Y}_k \otimes \mathcal{Z}_k), \tag{55}$$
such that the channel $\Xi_n \in \mathrm{C}(\mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n, \mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_n)$ defined as
$$\Xi_n = (\mathbb{1}_{\mathrm{L}(\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_{n-1})} \otimes \Phi_n) \cdots (\Phi_1 \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X}_2 \otimes \cdots \otimes \mathcal{X}_n)}) \tag{56}$$
satisfies
$$X = (\Xi_n \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n)})(\mathrm{vec}(\mathbb{1}_{\mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n})\,\mathrm{vec}(\mathbb{1}_{\mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n})^*). \tag{57}$$
3. There exist complex Euclidean spaces $\mathcal{Z}_1, \ldots, \mathcal{Z}_{n-1}$ (and $\mathcal{Z}_0 = \mathbb{C}$ and $\mathcal{Z}_n = \mathbb{C}$), along with completely positive and unital maps $\Psi_1, \ldots, \Psi_n$ having the form
$$\Psi_k \in \mathrm{C}(\mathcal{Y}_k \otimes \mathcal{Z}_k, \mathcal{Z}_{k-1} \otimes \mathcal{X}_k), \tag{58}$$
such that the unital map $\Lambda_n \in \mathrm{CP}(\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_n, \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n)$ defined as
$$\Lambda_n = (\Psi_1 \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X}_2 \otimes \cdots \otimes \mathcal{X}_n)}) \cdots (\mathbb{1}_{\mathrm{L}(\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_{n-1})} \otimes \Psi_n) \tag{59}$$
satisfies
$$X = (\mathbb{1}_{\mathrm{L}(\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_n)} \otimes \Lambda_n)(\mathrm{vec}(\mathbb{1}_{\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_n})\,\mathrm{vec}(\mathbb{1}_{\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_n})^*). \tag{60}$$

Through this equivalence, for a given state $\rho \in \mathrm{D}(\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_n \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n)$, one arrives at an alternative interpretation of the semidefinite program

$$\begin{aligned} \text{maximize:} \quad & \langle \rho, X \rangle \\ \text{subject to:} \quad & X \in \mathcal{S}_n(\mathcal{X}_1, \ldots, \mathcal{X}_n; \mathcal{Y}_1, \ldots, \mathcal{Y}_n) \end{aligned} \tag{61}$$

that concerns an online variant of entanglement manipulation, as is explained shortly. The term "online" in this context refers to a situation in which a quantum state must be manipulated in multiple turns, where an output is required immediately after each input system arrives and prior to the next input system being made available, similar to an online process.

By the equivalence of the third statement above to the first, a maximization over all $X \in \mathcal{S}_n(\mathcal{X}_1, \ldots, \mathcal{X}_n; \mathcal{Y}_1, \ldots, \mathcal{Y}_n)$ is equivalent to a maximization over all operators

$$(\mathbb{1}_{\mathrm{L}(\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_n)} \otimes \Lambda_n)(\operatorname{vec}(\mathbb{1}_{\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_n}) \operatorname{vec}(\mathbb{1}_{\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_n})^*) \tag{62}$$

for

$$\Lambda_n = (\Psi_1 \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X}_2 \otimes \cdots \otimes \mathcal{X}_n)}) \cdots (\mathbb{1}_{\mathrm{L}(\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_{n-1})} \otimes \Psi_n) \tag{63}$$

and $\Psi_1, \ldots, \Psi_n$ being completely positive and unital maps of the form

$$\Psi_k \in \mathrm{C}(\mathcal{Y}_k \otimes \mathcal{Z}_k, \mathcal{Z}_{k-1} \otimes \mathcal{X}_k). \tag{64}$$

The value of the objective function $\langle \rho, X \rangle$ may therefore be expressed as

$$\langle (\mathbb{1}_{\mathrm{L}(\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_n)} \otimes \Lambda_n^*)(\rho), \operatorname{vec}(\mathbb{1}_{\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_n}) \operatorname{vec}(\mathbb{1}_{\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_n})^* \rangle, \tag{65}$$

which is $\dim(\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_n)$ times the squared fidelity between the maximally entangled state $\tau \in \mathrm{D}(\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_n \otimes \mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_n)$ given by

$$\tau = \frac{\operatorname{vec}(\mathbb{1}_{\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_n}) \operatorname{vec}(\mathbb{1}_{\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_n})^*}{\dim(\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_n)} \tag{66}$$

and the state obtained by applying the channel Λ_n^* to the portion of ρ corresponding to the spaces $\mathcal{X}_1, \ldots, \mathcal{X}_n$. In the case that $n = 1$, König, Renner, and Schaffner [12] refer to this quantity as the *quantum correlation*. This situation is illustrated for the case $n = 3$ in Figure 8.

By Theorem 1, one finds that when ρ is pure, the same optimal value is achieved when the ordering of the channels and the registers on which they act is reversed, as illustrated in Figure 9 for the case $n = 3$. That is, when ρ is a pure state, the optimal value of the semidefinite program (61) represents the value

$$\langle (\Xi_n \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n)})(\rho), \operatorname{vec}(\mathbb{1}_{\mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n}) \operatorname{vec}(\mathbb{1}_{\mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n})^* \rangle, \tag{67}$$

maximized over all channels $\Xi_n \in \mathrm{C}(\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_n, \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n)$ of the form

$$\Xi_n = (\Phi_1 \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X}_2 \otimes \cdots \otimes \mathcal{X}_n)}) \cdots (\mathbb{1}_{\mathrm{L}(\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_{n-1})} \otimes \Phi_n) \tag{68}$$

for channels $\Phi_1, \ldots, \Phi_n$ taking the form

$$\Phi_k \in \mathrm{C}(\mathcal{Y}_k \otimes \mathcal{Z}_k, \mathcal{Z}_{k-1} \otimes \mathcal{X}_k) \tag{69}$$

and for $\mathcal{Z}_2, \ldots, \mathcal{Z}_{n-1}$ arbitrary complex Euclidean spaces (along with $\mathcal{Z}_0 = \mathbb{C}$ and $\mathcal{Z}_n = \mathbb{C}$).

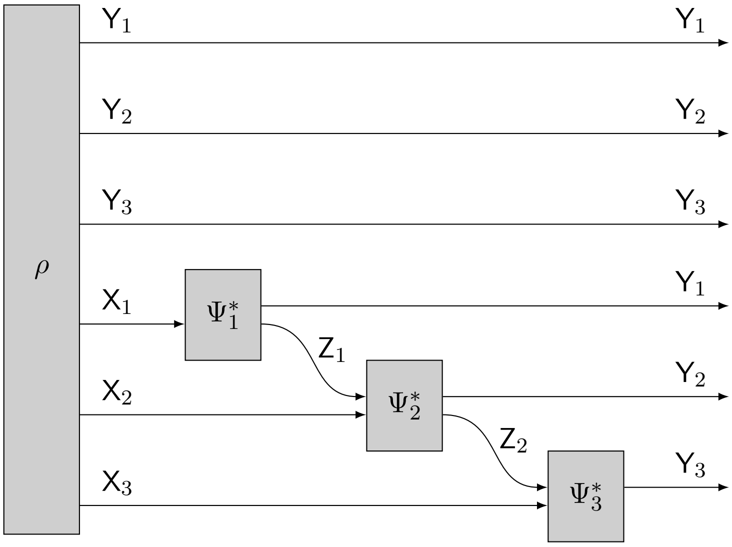

Figure 8: The channel $\Lambda_3^* = (\mathbb{1}_{\mathrm{L}(\mathcal{Y}_1 \otimes \mathcal{Y}_2)} \otimes \Psi_3^*)(\mathbb{1}_{\mathrm{L}(\mathcal{Y}_1)} \otimes \Psi_2^* \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X}_3)})(\Psi_1^* \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X}_2 \otimes \mathcal{X}_3)})$ is applied to registers $(\mathsf{X}_1, \mathsf{X}_2, \mathsf{X}_3)$ of a state $\rho \in \mathrm{D}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 \otimes \mathcal{Y}_3 \otimes \mathcal{X}_1 \otimes \mathcal{X}_2 \otimes \mathcal{X}_3)$ with the aim of maximizing the fidelity of the output state with the canonical maximally entangled state.

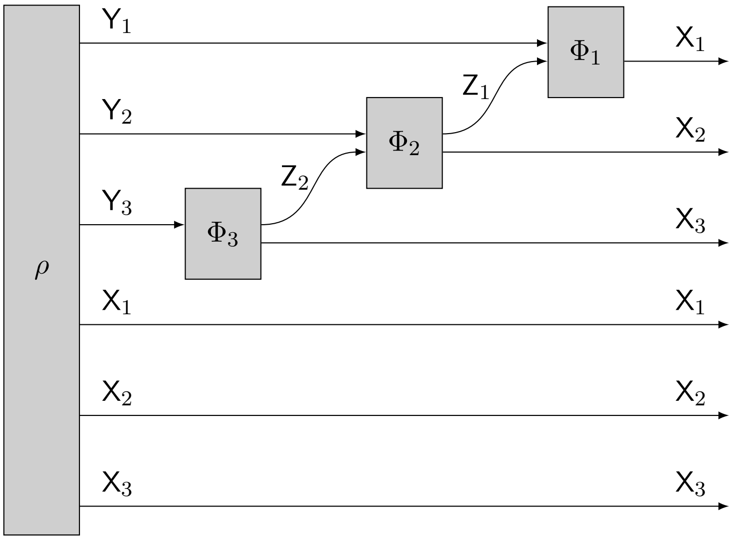

Figure 9: A similar process to the one illustrated in Figure 8, but with channels applied to Y_3, Y_2, Y_1 rather than X_1, X_2, X_3.

5 Conclusion

We have identified a time-reversal property for rank-one quantum strategy functions, explained its connection to conditional min- and max-entropy, and described an alternative view of this property through an online variant of pure state entanglement manipulation. An obvious question arises: are there interesting applications or implications of this property beyond those we have mentioned?

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