Lecture 7

A semidefinite program for the entangled bias of XOR games

In this lecture we will discuss a couple of applications of Tsirelson's theorem, centering primarily on the semidefinite programming formulation for the entangled bias of XOR games that it yields.

7.1 The semidefinite program

Let us begin with the semidefinite program that is suggested by Tsirelson's theorem. Assume that $G = (X, Y, \pi, f)$ is an XOR game, and recall (as was discussed in the previous lecture) that the entangled bias of G is the supremum of the values

$$\sum_{(x,y)\in X\times Y} \pi(x,y)(-1)^{f(x,y)}\langle A_x \otimes B_y, \rho\rangle, \tag{7.1}$$

taken over all choices for complex Euclidean spaces $\mathcal{A}$ and $\mathcal{B}$, a state $\rho \in \mathrm{D}(\mathcal{A}\otimes\mathcal{B})$, and Hermitian contractions

$$\{A_x : x \in X\} \subset \mathrm{Herm}(\mathcal{A}) \quad \text{and} \quad \{B_y : y \in Y\} \subset \mathrm{Herm}(\mathcal{B}). \tag{7.2}$$

By Tsirelson's theorem, this is equivalent to the supremum of the values

$$\sum_{(x,y)\in X\times Y} \pi(x,y)(-1)^{f(x,y)} M(x,y), \tag{7.3}$$

taken over all $M \in \mathrm{L}(\mathbb{R}^Y, \mathbb{R}^X)$ for which there exist $R \in \mathrm{Pos}(\mathbb{C}^X)$ and $S \in \mathrm{Pos}(\mathbb{C}^Y)$ for which $R(x,x) = 1$ for all $x \in X$, $S(y,y) = 1$ for all $y \in Y$, and

$$\begin{pmatrix} R & M \\ M^* & S \end{pmatrix} \in \mathrm{Pos}(\mathbb{C}^X \oplus \mathbb{C}^Y). \tag{7.4}$$

With this fact in mind, let us consider the following semidefinite program. First, let us write $\mathcal{X} = \mathbb{C}^X$ and $\mathcal{Y} = \mathbb{C}^Y$ for brevity, and let $\Delta \in \mathrm{C}(\mathcal{X} \oplus \mathcal{Y})$ denote the completely dephasing channel acting on $\mathcal{X} \oplus \mathcal{Y}$, which zeros out all of the off-diagonal entries of its input and leaves the diagonal entries alone. Define an operator $D \in \mathrm{L}(\mathcal{Y}, \mathcal{X})$ as

$$D(x, y) = \pi(x, y)(-1)^{f(x,y)} \tag{7.5}$$

for every $x \in X$ and $y \in Y$, and let $H \in \mathrm{Herm}(\mathcal{X} \oplus \mathcal{Y})$ be defined as

$$H = \frac{1}{2}\begin{pmatrix} 0 & D \\ D^* & 0 \end{pmatrix}. \tag{7.6}$$

The semidefinite program to be considered is described by the triple $(\Delta, H, \mathbb{1}_{\mathcal{X}\oplus\mathcal{Y}})$. The primal and dual problems associated with this semidefinite program take the following form.

Optimization Problem 7.1 (SDP for XOR game bias, unsimplified)

Primal problem

$$\begin{aligned} \text{maximize:}\quad & \langle H, Z\rangle \\ \text{subject to:}\quad & \Delta(Z) = \mathbb{1}_{\mathcal{X}\oplus\mathcal{Y}}, \\ & Z \in \mathrm{Pos}(\mathcal{X} \oplus \mathcal{Y}). \end{aligned}$$

Dual problem

$$\begin{aligned} \text{minimize:}\quad & \mathrm{Tr}(W) \\ \text{subject to:}\quad & \Delta(W) \geq H, \\ & W \in \mathrm{Herm}(\mathcal{X} \oplus \mathcal{Y}). \end{aligned}$$

Note that in the dual problem formulation we have used the fact that the completely dephasing channel is self-dual: $\Delta = \Delta^*$. Strong duality and the achievability of the optimal values in both the primal and dual problems follow from Slater's theorem; strictly feasible solutions are given by $Z = \mathbb{1}_{\mathcal{X}\oplus\mathcal{Y}}$ in the primal and $W = \lambda\mathbb{1}_{\mathcal{X}\oplus\mathcal{Y}}$ for a sufficiently large λ in the dual.

Let us now examine both the primal and dual problems, beginning with the primal problem. Our principal order of business with the primal problem, which is fairly straightforward, is to verify that its optimal value indeed agrees with the entangled bias of the XOR game G.

Suppose first that $M \in \mathrm{L}(\mathbb{R}^Y, \mathbb{R}^X)$ is such that there exist $R \in \mathrm{Pos}(\mathcal{X})$ and $S \in \mathrm{Pos}(\mathcal{Y})$ for which $R(x, x) = 1$ for all $x \in X$, $S(y, y) = 1$ for all $y \in Y$, and

$$Z = \begin{pmatrix} R & M \\ M^* & S \end{pmatrix} \in \mathrm{Pos}(\mathcal{X} \oplus \mathcal{Y}). \tag{7.7}$$

The operator Z is then primal feasible, as the constraint $\Delta(Z) = \mathbb{1}_{\mathcal{X}\oplus\mathcal{Y}}$ is equivalent to R and S having diagonal entries equal to one. The objective value for Z is equal to

$$\langle H, Z\rangle = \frac{1}{2}\langle D, M\rangle + \frac{1}{2}\langle D^*, M^*\rangle = \langle D, M\rangle, \tag{7.8}$$

owing to the fact that D and M have real number entries. Noting that

$$\langle D, M \rangle = \sum_{(x,y)\in X\times Y} \pi(x,y)(-1)^{f(x,y)} M(x,y), \tag{7.9}$$

we find that the optimal value of the semidefinite program is at least the entangled bias of G.

To see that the optimal value of the semidefinite program is no greater than the entangled bias of G, consider any $Z \in \mathrm{Pos}(\mathcal{X} \oplus \mathcal{Y})$, which can be expressed as

$$Z = \begin{pmatrix} R & K \\ K^* & S \end{pmatrix} \tag{7.10}$$

for some choice of $R \in \mathrm{Pos}(\mathcal{X})$, $S \in \mathrm{Pos}(\mathcal{Y})$, and $K \in \mathrm{L}(\mathcal{Y}, \mathcal{X})$. Again the constraint $\Delta(Z) = \mathbb{1}_{\mathcal{X}\oplus\mathcal{Y}}$ is equivalent to R and S having diagonal entries equal to one, and the only issue remaining is that K might not have real number entries. However, by expressing the objective function in terms of the block structure of H and Z, we find that

$$\langle H, Z \rangle = \frac{1}{2}\langle D, K \rangle + \frac{1}{2}\langle D^*, K^* \rangle = \langle D, M \rangle \tag{7.11}$$

for

$$M = \frac{K + \overline{K}}{2}, \tag{7.12}$$

following from the fact that D has real entries:

$$\langle D^*, K^* \rangle = \overline{\langle D, K \rangle} = \langle \overline{D}, \overline{K} \rangle = \langle D, \overline{K} \rangle. \tag{7.13}$$

Proceeding in much the same way as in the previous lecture, we see that

$$\frac{1}{2}\begin{pmatrix} R & K \\ K^* & S \end{pmatrix} + \frac{1}{2}\begin{pmatrix} R & K \\ K^* & S \end{pmatrix}^{\mathsf{T}} = \begin{pmatrix} \frac{1}{2}R + \frac{1}{2}R^{\mathsf{T}} & M \\ M^* & \frac{1}{2}S + \frac{1}{2}S^{\mathsf{T}} \end{pmatrix} \tag{7.14}$$

is positive semidefinite, and the diagonal entries of $\frac{1}{2}R + \frac{1}{2}R^{\mathsf{T}}$ and $\frac{1}{2}S + \frac{1}{2}S^{\mathsf{T}}$, are all equal to one, and therefore the objective value $\langle D, M \rangle$ is no greater than the entangled bias of G.

Now let us consider the dual problem, in which one maximizes $\mathrm{Tr}(W)$ over all $W \in \mathrm{Herm}(\mathcal{X} \oplus \mathcal{Y})$, subject to the constraint that

$$\Delta(W) \geq H. \tag{7.15}$$

Notice that the off-diagonal entries of W have absolutely no influence on this problem: they are zeroed out by Δ in the constraint, and they do not influence the objective function. For this reason there is no generality lost in restricting one's attention to W taking the form

$$W = \frac{1}{2}\begin{pmatrix} \mathrm{Diag}(u) & 0 \\ 0 & \mathrm{Diag}(v) \end{pmatrix} \tag{7.16}$$

for vectors $u \in \mathbb{R}^X$ and $v \in \mathbb{R}^Y$. (Here we're including the factor of 1/2 for the sake of convenience—we're free to scale the vectors u and v as we choose.) The objective function then becomes

$$\operatorname{Tr}(W) = \frac{1}{2}\sum_{x\in X} u(x) + \frac{1}{2}\sum_{y\in Y} v(y), \tag{7.17}$$

while the constraint becomes equivalent to

$$\begin{pmatrix} \operatorname{Diag}(u) & -D \\ -D^* & \operatorname{Diag}(v) \end{pmatrix} \geq 0. \tag{7.18}$$

Summarizing what we have just concluded about Optimization Problem 7.1, we arrive at the following expression of the same problem.

Optimization Problem 7.2 (SDP for XOR game bias, simplified)

Primal problem

$$\begin{aligned} \text{maximize:} \quad & \langle D, M\rangle \\ \text{subject to:} \quad & \begin{pmatrix} R & M \\ M^* & S \end{pmatrix} \geq 0, \\ & R(x,x) = 1 \text{ for all } x \in X, \\ & S(y,y) = 1 \text{ for all } y \in Y, \\ & R \in \operatorname{Pos}(\mathcal{X}), \\ & S \in \operatorname{Pos}(\mathcal{Y}), \\ & M \in \operatorname{L}(\mathbb{R}^Y, \mathbb{R}^X). \end{aligned}$$

Dual problem

$$\begin{aligned} \text{minimize:} \quad & \frac{1}{2}\sum_{x\in X} u(x) + \frac{1}{2}\sum_{y\in Y} v(y) \\ \text{subject to:} \quad & \begin{pmatrix} \operatorname{Diag}(u) & -D \\ -D^* & \operatorname{Diag}(v) \end{pmatrix} \geq 0, \\ & u \in \mathbb{R}^X, \\ & v \in \mathbb{R}^Y. \end{aligned}$$

It will be helpful later in the lecture for us to observe at this point that for any dual-optimal choice of u and v, it must be the case that

$$\sum_{x\in X} u(x) = \sum_{y\in Y} v(y). \tag{7.19}$$

The reason is that for any choice of u and v, and for any $\lambda > 0$, the operator

$$\begin{pmatrix} \operatorname{Diag}(u) & -D \\ -D^* & \operatorname{Diag}(v) \end{pmatrix} \tag{7.20}$$

is positive semidefinite if and only if

$$\begin{pmatrix} \lambda \operatorname{Diag}(u) & -D \\ -D^* & \frac{1}{\lambda}\operatorname{Diag}(v) \end{pmatrix} \tag{7.21}$$

is positive semidefinite. The dual objective value obtained by the operator (7.21), assuming it is positive semidefinite, is equal to

$$\frac{\lambda}{2} \sum_{x \in X} u(x) + \frac{1}{2\lambda} \sum_{y \in Y} v(y). \tag{7.22}$$

Assuming that D is nonzero, which is always the case when it arises from an XOR game G, it must be the case that $\sum_{x \in X} u(x)$ and $\sum_{y \in Y} v(y)$ are strictly positive, and in this case the minimum value for (7.22) occurs when

$$\lambda = \sqrt{\frac{\sum_{y \in Y} v(y)}{\sum_{x \in X} u(x)}}, \tag{7.23}$$

and this is the unique choice of λ for which the minimum is obtained. Thus, under the assumption that u and v are optimal, it must be the case that $\lambda = 1$, which is equivalent to (7.19).

We can now verify that the entangled value of the CHSH game is $\cos^2(\pi/8)$, as claimed in the previous lecture.

Example 7.3 (CHSH game entangled bias/value)**.** Recall that the CHSH game is the XOR game $G = (X, Y, \pi, f)$ with

$$\begin{gathered} X = Y = \{0, 1\}, \\ \pi(0,0) = \pi(0,1) = \pi(1,0) = \pi(1,1) = \frac{1}{4}, \\ f(x,y) = x \wedge y. \end{gathered} \tag{7.24}$$

The matrix $D(x, y) = \pi(x, y)(-1)^{f(x,y)}$ is then equal to

$$D = \frac{1}{4} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{7.25}$$

We will verify that the optimal value of Optimization Problem 7.2 for the game G is $\varepsilon^*(G) = 1/\sqrt{2}$.

First choose

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{7.26}$$

which has spectral norm equal to 1—it is a unitary operator, representing the Hadamard transform—and observe that for $R = S = \mathbb{1}$ we have that

$$Z = \begin{pmatrix} R & M \\ M^* & S \end{pmatrix} \geq 0. \tag{7.27}$$

The diagonal entries of R and S are equal to one, so Z is primal feasible, and it achieves the objective value $\langle D, M \rangle = 1/\sqrt{2}$.

In the dual problem, choosing

$$u = \left(\tfrac{1}{2\sqrt{2}}, \tfrac{1}{2\sqrt{2}} \right) \quad \text{and} \quad v = \left(\tfrac{1}{2\sqrt{2}}, \tfrac{1}{2\sqrt{2}} \right) \tag{7.28}$$

yields a feasible solution. The objective value is the same value just achieved in the primal:

$$\frac{1}{2} \sum_{x \in \{0,1\}} u(x) + \frac{1}{2} \sum_{y \in \{0,1\}} v(y) = \frac{1}{\sqrt{2}}. \tag{7.29}$$

Having obtained the same value in the primal and dual, we have verified that the optimal value, which is the entangled bias, is

$$\varepsilon^*(G) = \frac{1}{\sqrt{2}}. \tag{7.30}$$

This implies that the entangled value of the CHSH game is

$$\omega^*(G) = \frac{1}{2} + \frac{1}{2\sqrt{2}} = \cos^2(\pi/8). \tag{7.31}$$

7.2 Strong parallel repetition for XOR games

Next we will prove that XOR games obey a *strong parallel repetition* property. To explain what this means, let us first discuss the notion of parallel repetition in greater generality.

Given arbitrary nonlocal games $G_1, \ldots, G_n$, described by probability distributions

$$\begin{aligned} \pi_1 &: X_1 \times Y_1 \rightarrow [0,1], \\ &\vdots \\ \pi_n &: X_n \times Y_n \rightarrow [0,1], \end{aligned} \tag{7.32}$$

and predicates

$$\begin{aligned} V_1 &: A_1 \times B_1 \times X_1 \times Y_1 \rightarrow \{0,1\}, \\ &\vdots \\ V_n &: A_n \times B_n \times X_n \times Y_n \rightarrow \{0,1\}, \end{aligned} \tag{7.33}$$

respectively, one defines the nonlocal game $G = G_1 \wedge \cdots \wedge G_n$ by the distribution

$$\pi((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \pi_1(x_1, y_1) \cdots \pi_n(x_n, y_n) \tag{7.34}$$

and the predicate

$$V((a_1,\ldots,a_n),(b_1,\ldots,b_n)|(x_1,\ldots,x_n),(y_1,\ldots,y_n)) \\ = V_1(a_1,b_1|x_1,y_1)\wedge\cdots\wedge V_n(a_n,b_n|x_n,y_n). \tag{7.35}$$

In words, the game G is run as if it were independent instances of the games $G_1,\ldots,G_n$, where Alice and Bob receive n-tuples of questions $(x_1,\ldots,x_n)$ and $(y_1,\ldots,y_n)$ all at the same time, with each question pair (x_k,y_k) being chosen according to π_k, independent of all other question pairs. They are expected to provide answers $(a_1,\ldots,a_n)$ and $(b_1,\ldots,b_n)$, respectively, and they win the game G if and only if every one of the pairs (a_k,b_k) is correct for the corresponding question pair (x_k,y_k) in the game G_k.

It is important to realize, though, that Alice and Bob are not required to treat the individual games $G_1,\ldots,G_n$ independently. They may, in particular, attempt to correlate their answers in otherwise independent game instances to their advantage. The following example illustrates that it is indeed possible for them to gain an advantage along these lines.

Example 7.4 (Parallel repetition of the FFL game)**.** The classical value of the FFL game, introduced in the previous lecture, is 2/3. Remarkably, the classical value of the two-fold repetition FFL ∧ FFL of this game with itself is also 2/3. A deterministic strategy that achieves this winning probability is that Alice and Bob both respond to their question pairs by simply swapping the two binary values. That is, Alice's answers are determined by the function $f : X\times X\rightarrow A\times A$ and Bob's answers are determined by the function $g : Y\times Y\rightarrow B\times B$, where

$$f(x_1,x_2)=(x_2,x_1)\quad\text{and}\quad g(y_1,y_2)=(y_2,y_1). \tag{7.36}$$

For this strategy, the winning condition in both games is the same:

$$x_1\vee x_2\neq y_1\vee y_2. \tag{7.37}$$

That is, they either win both games or lose both games, never winning just one of them. If (x_1,y_1) and (x_2,y_2) are independently and uniformly generated from the set $\{(0,0),(0,1),(1,0)\}$ then the above condition fails only when

$$((x_1,y_1),(x_2,y_2))\in\big\{((0,0),(0,0)),\ ((1,0),(0,1)),\ ((0,1),(1,0))\big\}. \tag{7.38}$$

Such question pairs are selected with probability 3/9 = 1/3, so the winning probability is 2/3, as claimed.

The example of the FFL game illustrates that the classical value of a nonlocal game $G = G_1\wedge\cdots\wedge G_n$ is not always equal to the product of the values of

$G_1, \ldots, G_n$, which is what one obtains when Alice and Bob play the games independently. That is, one always has

$$\omega(G_1 \wedge \cdots \wedge G_n) \geq \omega(G_1) \cdots \omega(G_n), \tag{7.39}$$

but in some cases the inequality is strict. A similar phenomenon occurs for the entangled value—although we did not prove it, the entangled value of the FFL game agrees with the classical value, and therefore

$$\omega^*(\mathrm{FFL} \wedge \mathrm{FFL}) \geq \omega(\mathrm{FFL} \wedge \mathrm{FFL}) = \frac{2}{3} = \omega^*(\mathrm{FFL}) > \omega^*(\mathrm{FFL})^2. \tag{7.40}$$

We will prove that the entangled value of XOR games forbids this type of advantage. That is, if $G_1, \ldots, G_n$ are XOR games, then

$$\omega^*(G_1 \wedge \cdots \wedge G_n) = \omega^*(G_1) \cdots \omega^*(G_n). \tag{7.41}$$

This is the property referred to as *strong parallel repetition*. In particular, if G is an XOR game and we write

$$G^{\wedge n} = G \wedge \cdots \wedge G \quad (n \text{ times}), \tag{7.42}$$

then it necessarily holds that

$$\omega^*(G^{\wedge n}) = \omega^*(G)^n. \tag{7.43}$$

The first step in proving that (7.41) holds for XOR games $G_1, \ldots, G_n$ is to define the XOR of two (or more) XOR games. Suppose that G_1 and G_2 are XOR games, specified by probability distributions

$$\pi_1 : X_1 \times Y_1 \to [0,1] \quad \text{and} \quad \pi_2 : X_2 \times Y_2 \to [0,1] \tag{7.44}$$

along with functions

$$f_1 : X_1 \times Y_1 \to \{0,1\} \quad \text{and} \quad f_2 : X_2 \times Y_2 \to \{0,1\}. \tag{7.45}$$

The XOR $G_1 \oplus G_2$ of these two games is the XOR game defined by the distribution $\pi : (X_1 \times X_2) \times (Y_1 \times Y_2) \to [0,1]$ given by

$$\pi((x_1, x_2), (y_1, y_2)) = \pi_1(x_1, y_1)\pi_2(x_2, y_2) \tag{7.46}$$

and the function $f : (X_1 \times X_2) \times (Y_1 \times Y_2) \to \{0,1\}$ given by

$$f((x_1, x_2), (y_1, y_2)) = f_1(x_1, y_1) \oplus f_2(x_2, y_2). \tag{7.47}$$

In words, the question sets for the game $G_1 \oplus G_2$ are $X_1 \times X_2$ and $Y_1 \times Y_2$, and if Alice receives (x_1, x_2) and Bob receives (y_1, y_2), they are expected to provide answers $a, b \in \{0, 1\}$ that are consistent with the equations

$$a = a_1 \oplus a_2 \quad \text{and} \quad b = b_1 \oplus b_2 \tag{7.48}$$

for some choice of $a_1, a_2, b_1, b_2 \in \{0, 1\}$ that would cause (a_1, b_1) to be correct for (x_1, y_1) and (a_2, b_2) to be correct for (x_2, y_2). The XOR of more than two XOR games is defined similarly (or, equivalently, by applying this definition iteratively).

It is evident, for any two XOR games G_1 and G_2, that

$$\varepsilon^*(G_1 \oplus G_2) \geq \varepsilon^*(G_1)\varepsilon^*(G_2). \tag{7.49}$$

The reason is that if Alice and Bob play the game $G_1 \oplus G_2$ by playing G_1 and G_2 independently and optimally, and then answer according to the XOR of their answers in G_1 and G_2, then they will achieve the bias $\varepsilon^*(G_1)\varepsilon^*(G_2)$. We will prove that this is, in fact, the best they can do. That is, we will prove

$$\varepsilon^*(G_1 \oplus G_2) = \varepsilon^*(G_1)\varepsilon^*(G_2). \tag{7.50}$$

This will be done using semidefinite programming duality.

In particular, consider the dual form of Optimization Problem 7.2, the semidefinite program for the entangled bias of an XOR game, for the two separate XOR games G_1 and G_2. Suppose that (u_1, v_1) and (u_2, v_2) represent dual-optimal solutions to these semidefinite programs. As argued previously, this implies

$$\sum_{x_1} u_1(x_1) = \sum_{y_1} v_1(y_1) \quad \text{and} \quad \sum_{x_2} u_2(x_2) = \sum_{y_2} v_2(y_2). \tag{7.51}$$

The dual form of Optimization Problem 7.2 for the entangled bias of the XOR game $G_1 \oplus G_2$ has the following form:

$$\begin{aligned}
\text{minimize:} \quad & \frac{1}{2} \sum_{x \in X_1 \times X_2} u(x) + \frac{1}{2} \sum_{y \in Y_1 \times Y_2} v(y) \\
\text{subject to:} \quad & \begin{pmatrix} \operatorname{Diag}(u) & -D_1 \otimes D_2 \\ -D_1^* \otimes D_2^* & \operatorname{Diag}(v) \end{pmatrix} \geq 0, \\
& u \in \mathbb{R}^{X_1 \times X_2}, \ v \in \mathbb{R}^{Y_1 \times Y_2}.
\end{aligned}$$

Here, D_1 and D_2 are the operators given by

$$\begin{aligned}
D_1(x_1, y_1) &= \pi_1(x_1, y_1)(-1)^{f_1(x_1, y_1)}, \\
D_2(x_2, y_2) &= \pi_2(x_2, y_2)(-1)^{f_2(x_2, y_2)}.
\end{aligned} \tag{7.52}$$

Next we observe that $u = u_1 \otimes u_2$ and $v = v_1 \otimes v_2$ provide a dual feasible solution to Optimization Problem 7.2 for $\varepsilon^*(G_1 \oplus G_2)$. To prove that this is so, it is helpful to observe that if

$$\begin{pmatrix} P_1 & X_1 \\ X_1^* & Q_1 \end{pmatrix} \geq 0 \quad \text{and} \quad \begin{pmatrix} P_2 & X_2 \\ X_2^* & Q_2 \end{pmatrix} \geq 0 \tag{7.53}$$

then

$$\begin{pmatrix} P_1 \otimes P_2 & X_1 \otimes X_2 \\ X_1^* \otimes X_2^* & Q_1 \otimes Q_2 \end{pmatrix} \geq 0. \tag{7.54}$$

This is so because, after some simultaneous re-ordering of rows and columns, the matrix above is a principal submatrix of the positive semidefinite matrix

$$\begin{pmatrix} P_1 & X_1 \\ X_1^* & Q_1 \end{pmatrix} \otimes \begin{pmatrix} P_2 & X_2 \\ X_2^* & Q_2 \end{pmatrix}. \tag{7.55}$$

Alternatively, by (7.53) we may conclude that

$$X_1 = \sqrt{P_1} K_1 \sqrt{Q_1} \quad \text{and} \quad X_2 = \sqrt{P_2} K_2 \sqrt{Q_2} \tag{7.56}$$

for K_1 and K_2 satisfying $\|K_1\| \leq 1$ and $\|K_2\| \leq 1$. As we therefore have

$$X_1 \otimes X_2 = \sqrt{P_1 \otimes P_2}(K_1 \otimes K_2)\sqrt{Q_1 \otimes Q_2}, \tag{7.57}$$

and $\|K_1 \otimes K_2\| = \|K_1\|\|K_2\| \leq 1$, it follows that (7.54) holds. One may therefore conclude that

$$\begin{pmatrix} \operatorname{Diag}(u_1) \otimes \operatorname{Diag}(u_2) & D_1 \otimes D_2 \\ D_1^* \otimes D_2^* & \operatorname{Diag}(v_1) \otimes \operatorname{Diag}(v_2) \end{pmatrix} \geq 0. \tag{7.58}$$

As $\operatorname{Diag}(u_1) \otimes \operatorname{Diag}(u_2) = \operatorname{Diag}(u)$ and $\operatorname{Diag}(v_1) \otimes \operatorname{Diag}(v_2) = \operatorname{Diag}(v)$, we have

$$\begin{aligned} &\begin{pmatrix} \operatorname{Diag}(u) & -D_1 \otimes D_2 \\ -D_1^* \otimes D_2^* & \operatorname{Diag}(v) \end{pmatrix} \\ &= \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \begin{pmatrix} \operatorname{Diag}(u) & D_1 \otimes D_2 \\ D_1^* \otimes D_2^* & \operatorname{Diag}(v) \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \geq 0. \end{aligned} \tag{7.59}$$

The objective value of the dual solution (u, v) is

$$\begin{aligned} \frac{1}{2}\sum_x u(x) + \frac{1}{2}\sum_y v(y) &= \frac{1}{2}\sum_{x_1} u_1(x_1) \sum_{x_2} u_2(x_2) + \frac{1}{2}\sum_{y_1} v_1(y_1) \sum_{y_2} v_2(y_2) \\ &= \frac{1}{2}\varepsilon^*(G_1)\,\varepsilon^*(G_2) + \frac{1}{2}\varepsilon^*(G_1)\,\varepsilon^*(G_2) = \varepsilon^*(G_1)\,\varepsilon^*(G_2), \end{aligned} \tag{7.60}$$

where we have used the fact that

$$\sum_{x_1} u_1(x_1) = \sum_{y_1} v_1(y_1) = \varepsilon^*(G_1) \quad \text{and} \quad \sum_{x_2} u_2(x_2) = \sum_{y_2} v_2(y_2) = \varepsilon^*(G_2). \tag{7.61}$$

Therefore, $\varepsilon^*(G_1 \oplus G_2) \leq \varepsilon^*(G_1)\varepsilon^*(G_2)$, which implies (7.50).

We are now prepared to prove that, for XOR games $G_1, \ldots, G_n$, it holds that

$$\omega^*(G_1 \wedge \cdots \wedge G_n) = \omega^*(G_1) \cdots \omega^*(G_n). \tag{7.62}$$

As we have already observed, it holds that

$$\omega^*(G_1 \wedge \cdots \wedge G_n) \geq \omega^*(G_1) \cdots \omega^*(G_n), \tag{7.63}$$

and therefore it remains to prove

$$\omega^*(G_1 \wedge \cdots \wedge G_n) \leq \omega^*(G_1) \cdots \omega^*(G_n), \tag{7.64}$$

Assume hereafter that XOR games $G_1, \ldots, G_n$ have been fixed, and consider an arbitrary strategy for Alice and Bob in the game $G_1 \wedge \cdots \wedge G_n$, through which Alice and Bob answer question tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ with answer tuples $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$, respectively. We will consider how well this strategy performs for the XOR game

$$G_{k_1} \oplus \cdots \oplus G_{k_m}, \tag{7.65}$$

for various choices of a subset $S = \{k_1, \ldots, k_m\} \subseteq \{1, \ldots, n\}$, provided that we define Alice and Bob's answers as

$$a_{k_1} \oplus \cdots \oplus a_{k_m} \quad \text{and} \quad b_{k_1} \oplus \cdots \oplus b_{k_m} \tag{7.66}$$

and where we assume that they have chosen to share randomly generated question pairs (x_k, y_k) for those choices of $k \notin S$.

To do this we will define binary-valued random variables $Z_1, \ldots, Z_n$ as

$$Z_k = a_k \oplus b_k \oplus f_k(x_k, y_k), \tag{7.67}$$

where we view $x_1, \ldots, x_n$, $y_1, \ldots, y_n$, $a_1, \ldots, a_n$, and $b_1, \ldots, b_n$ as random variables, with $(x_1, y_1), \ldots, (x_n, y_n)$ distributed independently according to the distributions $\pi_1, \ldots, \pi_n$ given by the games $G_1, \ldots, G_n$ and $a_1, \ldots, a_n, b_1, \ldots, b_n$ distributed and correlated with $x_1, \ldots, x_n, y_1, \ldots, y_n$ in whatever manner Alice and Bob's strategy determines. It holds that

$$Z_k = \begin{cases} 0 & \text{if Alice and Bob win } G_k \\ 1 & \text{if Alice and Bob lose } G_k \end{cases} \tag{7.68}$$

and therefore the probability of winning minus losing game G_k is equal to the expectation

$$\mathrm{E}\Big((-1)^{Z_k}\Big). \tag{7.69}$$

More generally, if Alice and Bob's strategy is transformed into a strategy for the XOR game $G_{k_1} \oplus \cdots \oplus G_{k_m}$ as suggested above, we find that

$$Z_{k_1} \oplus \cdots \oplus Z_{k_m} = \begin{cases} 0 & \text{if Alice and Bob win } G_{k_1} \oplus \cdots \oplus G_{k_m} \\ 1 & \text{if Alice and Bob lose } G_{k_1} \oplus \cdots \oplus G_{k_m} \end{cases} \tag{7.70}$$

and therefore the probability they win minus the probability they lose in this XOR game is

$$\mathrm{E}\Big((-1)^{Z_{k_1}+\cdots+Z_{k_m}}\Big). \tag{7.71}$$

Now, we know that probability of winning minus the probability of losing in the game $G_{k_1} \oplus \cdots \oplus G_{k_m}$ is upper-bounded by its bias:

$$\mathrm{E}\Big((-1)^{Z_{k_1}+\cdots+Z_{k_m}}\Big) \leq \varepsilon^*(G_{k_1} \oplus \cdots \oplus G_{k_m}) = \varepsilon^*(G_{k_1}) \cdots \varepsilon^*(G_{k_m}). \tag{7.72}$$

Here we have used the fact concerning XOR game biases proved above, which is the key to making the entire argument work. The probability that Alice and Bob's strategy wins $G_1 \wedge \cdots \wedge G_n$ is therefore bounded as follows:

$$\begin{aligned} \Pr(Z_1 = 0, \ldots, Z_n = 0) &= \mathrm{E}\bigg(\bigg(\frac{1+(-1)^{Z_1}}{2}\bigg) \cdots \bigg(\frac{1+(-1)^{Z_n}}{2}\bigg)\bigg) \\ &= \frac{1}{2^n} \sum_{S \subseteq \{1,\ldots,n\}} \mathrm{E}\Big((-1)^{\sum_{k\in S} Z_k}\Big) \\ &\leq \frac{1}{2^n} \sum_{S \subseteq \{1,\ldots,n\}} \prod_{k\in S} \varepsilon^*(G_k) \\ &= \bigg(\frac{1+\varepsilon^*(G_1)}{2}\bigg) \cdots \bigg(\frac{1+\varepsilon^*(G_n)}{2}\bigg) \\ &= \omega^*(G_1) \cdots \omega^*(G_n). \end{aligned} \tag{7.73}$$

Maximizing over all possible entangled strategies for Alice and Bob yields

$$\omega^*(G_1 \wedge \cdots \wedge G_n) \leq \omega^*(G_1) \cdots \omega^*(G_n), \tag{7.74}$$

as required.