Lecture 20: Bell inequalities and nonlocality

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So far in the course we have considered uses for quantum information in the settings of computation and cryptography. In this lecture and the next we will consider uses of quantum information in distributive settings. This lecture will focus on using entanglement to achieve stronger correlations than are possible classically between two or more parties that cannot communicate. In the next lecture we will discuss communication complexity, which considers the communication costs for performing various tasks, and we will see that quantum information can sometimes give very large reductions in these costs.

Nonlocal games

We saw very early in the course that entanglement is a useful resource when combined with other resources such as in teleportation and super-dense coding. However, we know that entanglement alone cannot be used by directly for communication. For instance, if Alice and Bob share some entangled state $|\psi\rangle \in \mathcal{A} \otimes \mathcal{B}$, and Alice performs any measurement on her part of the state, the results are completely determined by her reduced state $\text{Tr}_{\mathcal{B}} |\psi\rangle \langle\psi|$ and are independent of any operation that Bob could perform on his half of the state. The situation is similar for Bob.

However, if both Alice and Bob perform measurements, then there can be interesting *correlations* between their measurement outcomes that would not be possible using classical information. These correlations can be studied by means of *nonlocal games*.

In a nonlocal game there are two or more *players* (Alice, Bob, Charlie, Diane, . . .) who are viewed as cooperating with one another. A *referee* runs the game, and all communication in the game is between the players and the referee—no communication directly between any of the players is permitted. The referee randomly selects a *question* for each player and sends each question to the appropriate player. Each player sends an *answer* back to the referee, and based on all of the questions and answers the referee determines whether the players win or lose.

For example, if there are two players, a nonlocal game has the following structure:
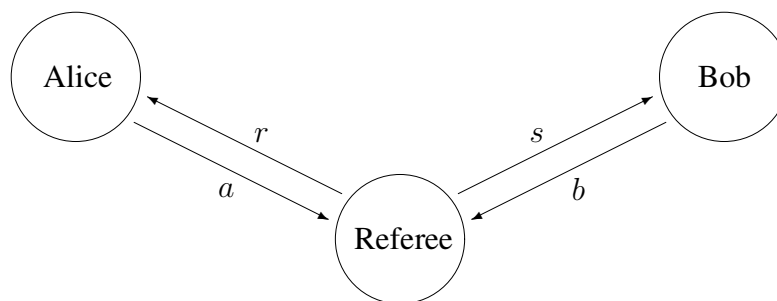
Here, the referee chooses a pair of questions (r, s) (according to some prespecified distribution), sends r to Alice and s to Bob, and Alice and Bob answer with a and b, respectively. The referee evaluates some predicate on (r, s, a, b) to determine if they win or lose.

We can speak of *classical strategies* where the players must behave classically or *quantum strategies* which allow the players to share an entangled state and to perform quantum measurements to determine their answers to the referees questions. What is interesting is that sometimes quantum strategies can allow the players to win with a much higher probability than they could with any classical strategy.

Example 1 (The GHZ game). Consider the following game, where there are three players: Alice, Bob, and Charlie. Each question and each answer will be a single bit.

The referee chooses a three bit string $r\,s\,t$ uniformly from the set $\{000, 011, 101, 110\}$ and sends r to Alice, s to Bob, and t to Charlie. Their answers must be bits: a from Alice, b from Bob, and c from Charlie. They win if $a \oplus b \oplus c = r \vee s \vee t$ and lose otherwise. The following table lists the winning condition for each possible set of questions:

$r\,s\,t$	$a \oplus b \oplus c$
000	0
011	1
101	1
110	1

What is the maximum probability of winning if Alice, Bob, and Charlie use a classical strategy? First consider a *deterministic strategy*, where each answer is a function of the question received and no randomness is used by the players. Let us write a_r, b_s and c_t to denote the answers that would be given for each choice of r, s, and t. For example, if $a_0 = 1$ and $a_1 = 0$, then Alice always answers the question 0 with 1 and the question 1 with 0. The winning conditions can be expressed by the four equations

$$a_0 \oplus b_0 \oplus c_0 = 0$$
$$a_0 \oplus b_1 \oplus c_1 = 1$$
$$a_1 \oplus b_0 \oplus c_1 = 1$$
$$a_1 \oplus b_1 \oplus c_0 = 1$$

Adding the four equations modulo 2 gives $0 = 1$, a contradiction. This means it is not possible for a deterministic strategy to win every time, so the probability of winning can be at most $3/4$ (because at least one of the four question sets will be answered incorrectly). It is easy to devise a strategy that wins $3/4$ of the time ($a_0 = a_1 = b_0 = b_1 = c_0 = c_1 = 1$, for instance), and so the maximum probability of winning is $3/4$.

What about probabilistic strategies? It turns out that for nonlocal games, probabilistic classical strategies can be no better than deterministic strategies. This is because the probability that a probabilistic strategy wins is just an average of the probabilities that some collection of deterministic strategies win. An average of a collection of numbers cannot be larger than the largest of the numbers.

Now let us turn to quantum strategies. Suppose that the three players share the entangled state

$$|\psi\rangle = \frac{1}{2}|000\rangle - \frac{1}{2}|011\rangle - \frac{1}{2}|101\rangle - \frac{1}{2}|110\rangle.$$

This is sometimes called a GHZ state—but also the term GHZ state sometimes refers to the state

$$\frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|111\rangle$$

which is equivalent to $|\psi\rangle$ up to local unitary operations. Each player will use the same strategy:

1. If the question is $q = 1$, then the player performs a Hadamard transform on their qubit of the above state. (If $q = 0$, the player does not perform a Hadamard transform.)

2. The player measures their qubit in the standard basis and returns the answer to the referee.

Alternately, the players measure with respect to the basis $\{|0\rangle, |1\rangle\}$ or $\{|+\rangle, |-\rangle\}$ depending on the question they received.

Let us see how well this strategy works. There are two cases:

Case 1: $r\,s\,t = 000$. In this case the players all just measure their qubit, and it is obvious that the results satisfy $a \oplus b \oplus c = 0$ as required.

Case 2: $r\,s\,t \in \{011, 101, 110\}$. All three possibilities will work the same way by symmetry, so let us assume $r\,s\,t = 011$. Notice that

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle \left(\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle\right) - \frac{1}{\sqrt{2}}|1\rangle \left(\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle\right)$$
$$= \frac{1}{\sqrt{2}}|0\rangle|\phi^-\rangle - \frac{1}{\sqrt{2}}|1\rangle|\psi^+\rangle.$$

Bob and Charlie both receive the question 1, so they both perform a Hadamard transform. It is easy to check that

$$(H \otimes H)|\phi^-\rangle = |\psi^+\rangle \qquad \text{and} \qquad (H \otimes H)|\psi^+\rangle = |\phi^-\rangle$$

so the state becomes

$$(I \otimes H \otimes H)|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle|\psi^+\rangle - \frac{1}{\sqrt{2}}|1\rangle|\phi^-\rangle = \frac{1}{2}(|001\rangle + |010\rangle - |100\rangle + |111\rangle).$$

When they measure, the results satisfy $a \oplus b \oplus c = 1$ as required. We have therefore shown that there is a quantum strategy that wins every time.

Remark. This example, as well as the others we will discuss, appear in the physics literature not as games but as hypothetical experiments. The game formulation is natural and appealing for theoretical computer scientists, as we are used to thinking about different parties optimizing their performance in various abstract settings. In physics, the notion of a classical strategy is typically

replaced by the notion of a *hidden variable theory*, and good quantum strategies simply coincide with the setting up of the experiment. Theoretically, one finds that the statistics of the experiment could not be predicted by a hidden variable theory.

Although the interpretation is critical in physics, the mathematics is the same in the two settings. Because our focus is on the mathematics, and this is a computer science course, I will stick mainly with the game formulation of these examples.

Example 2 (The CHSH game). The set-up for this game is similar to the GHZ game, except that there are only two players: Alice and Bob. The referee chooses questions $rs \in \{00, 01, 10, 11\}$ uniformly, and Alice and Bob must each answer a single bit: a for Alice, b for Bob. They win if $a \oplus b = r \wedge s$, i.e., the winning conditions are:

rs	$a \oplus b$
00	0
01	0
10	0
11	1

By similar reasoning to the GHZ game, the maximum probability with which a classical strategy can win is $3/4$.

There are two ways to describe the quantum strategy that we will consider. The first way will be more familiar, but the second way is useful and this is a good opportunity to learn something new.

Here is the first way. The entangled state shared by Alice and Bob will be $|\psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. For a given angle $\theta \in [0, 2\pi)$, define

$$|\phi_0(\theta)\rangle = \cos(\theta) |0\rangle + \sin(\theta) |1\rangle,$$

$$|\phi_1(\theta)\rangle = -\sin(\theta) |0\rangle + \cos(\theta) |1\rangle.$$

If Alice receives the question 0, she will measure her qubit with respect to the basis

$$\{|\phi_0(0)\rangle, |\phi_1(0)\rangle\},$$

and if she receives the question 1, she will measure her qubit with respect to the basis

$$\{|\phi_0(\pi/4)\rangle, |\phi_1(\pi/4)\rangle\}.$$

Bob uses a similar strategy, except that he measures with respect to the basis

$$\{|\phi_0(\pi/8)\rangle, |\phi_1(\pi/8)\rangle\} \quad \text{or} \quad \{|\phi_0(-\pi/8)\rangle, |\phi_1(-\pi/8)\rangle\},$$

depending on whether his question was 0 or 1.

The second way uses the notion of an *observable*. This term has different meanings to different people, but here is what I mean. Suppose we have a projective measurement $\{\Pi_a : a \in \Gamma\}$ where

the set Γ of measurement outcomes is finite (as usual) and is a subset of the real numbers: $\Gamma \subset \mathbb{R}$. Then the *observable* that corresponds to this measurement is a single Hermitian matrix:

$$A = \sum_{a \in \Gamma} a \Pi_a.$$

In other words, the eigenspaces correspond to the projection operators, and the eigenvalues correspond to the measurement outcomes. If a pure state $|\psi\rangle$ is measured with respect to this observable, then the *expected value* of the outcome is simply $\langle \psi | A | \psi \rangle$.

For instance, if we measure with respect to the standard basis and associate the outcome 1 with $|0\rangle$ and -1 with $|1\rangle$, then the corresponding observable is $|0\rangle \langle 0| - |1\rangle \langle 1| = \sigma_z$. A similar example using the diagonal basis give observable $|+\rangle \langle +| - |-\rangle \langle -| = \sigma_x$.

Now, back to the CHSH game, let us imagine that Alice and Bob both answer ± 1, where we are making the identification $a \leftrightarrow (-1)^a$. Then Alice's observables are as follows:

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z \qquad \text{and} \qquad A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x$$

while Bob's observables are

$$B_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = H \qquad \text{and} \qquad B_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} = ??.$$

(I don't know of any special name for B_1.) These observables simply come from translating the first description of the strategy to observables, using some very well known trigonometric inequalities along the way.

Okay, so how well does the strategy do? A boring case analysis will reveal that for all 4 possible question pairs, Alice and Bob will answer correctly with probability $\cos^2(\pi/8) \approx 0.85$, which is better than an optimal classical strategy that wins with probability $3/4$.

We can also use the observables to prove this. Consider the expression

$$\frac{1}{4} \langle \psi | A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1 | \psi \rangle .$$

I claim that this is the probability that Alice and Bob win minus the probability they lose. This comes from the fact that $\langle \psi | A_r \otimes B_s | \psi \rangle$ is the expected value of the product of Alice and Bob's ± 1 measurement outcomes. When $rs \in \{00, 01, 10\}$ this is the probability of winning minus the probability of losing on questions (r, s) because they win when their measurement outcomes are the same (have product 1) and lose when they are different (have product -1). In the last case $rs = 11$, they win when their measurement outcomes disagree (have product -1) and lose if they are the same (have product 1), so we put a minus sign in front of the term $A_1 \otimes B_1$ to reflect this. A quick check shows that

$$\langle \psi | A_0 \otimes B_0 | \psi \rangle = \langle \psi | A_0 \otimes B_1 | \psi \rangle = \langle \psi | A_1 \otimes B_0 | \psi \rangle = - \langle \psi | A_1 \otimes B_1 | \psi \rangle = \frac{1}{\sqrt{2}}$$

and so the probability of winning minus the probability of losing is $\frac{1}{\sqrt{2}}$. This means the probability of winning is

$$\frac{1}{2} + \frac{1}{2\sqrt{2}} = \cos^2(\pi/8).$$

5

Bell Inequalities

You may be wondering why the title of the lecture is "Bell inequalities and nonlocality". In the physical interpretation of nonlocal games, the bounds on classical strategies (or on local hidden variable theories) are typically known as *Bell Inequalities*, and quantum strategies (or experiments) that beat these bounds are said to *violate* a Bell inequality.

Typically you would see the Bell Inequality (known as the CHSH inequality) associated with the second example stated as:

$$A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 \leq 2.$$

You can probably convince yourself easily that if A_0, A_1, B_0 and B_1 are treated as ordinary variables taking values in $[-1, 1]$, then the inequality must hold. This is equivalent to saying that there is no classical strategy that wins the CHSH game with probability exceeding 3/4. Because there exist observables A_0, A_1, B_0 and B_1 with eigenvalues in $[-1, 1]$ and a state $|\psi\rangle$ such that

$$\langle\psi|A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1|\psi\rangle = 2\sqrt{2}$$

means that this Bell inequality is violated by the corresponding experiment. In both cases, the tensor product can be viewed as being implicit, or simply removed but the assumption that A_a and B_b commute for each choice of a and b.

Tsirelson's bound

You might wonder if it is possible to do better than $\cos^2(\pi/8)$ with a quantum strategy for the CHSH game. The answer is "no" due to Tsirelson's bound:

$$\langle\psi|A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1|\psi\rangle \leq 2\sqrt{2}$$

for any choice of observables A_0, A_1, B_0 and B_1 with eigenvalues in $[-1, 1]$ and any state $|\psi\rangle$. This includes the possibility that $|\psi\rangle$ is a state on many qubits, not just two. This can be shown as follows, using the fact that $\|A_0\|, \|A_1\|, \|B_0\|, \|B_1\| \leq 1$ along with some other simple properties of the norm of an operator:

$$\langle\psi|A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1|\psi\rangle$$
$$\leq \|(A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1)|\psi\rangle\|$$
$$\leq \|(A_0 \otimes (B_0 + B_1))|\psi\rangle\| + \|(A_1 \otimes (B_0 - B_1))|\psi\rangle\|$$
$$\leq \|(I \otimes B_0)|\psi\rangle + (I \otimes B_1)|\psi\rangle\| + \|(I \otimes B_0)|\psi\rangle - (I \otimes B_1)|\psi\rangle\|$$
$$= \||\phi_0\rangle + |\phi_1\rangle\| + \||\phi_0\rangle - |\phi_1\rangle\|$$

for $|\phi_b\rangle = (I \otimes B_b)|\psi\rangle$. Because $\||\phi_b\rangle\| \leq 1$ we have

$$\||\phi_0\rangle + |\phi_1\rangle\| + \||\phi_0\rangle - |\phi_1\rangle\| \leq \sqrt{2 + 2\Re\langle\phi_0|\phi_1\rangle} + \sqrt{2 - 2\Re\langle\phi_0|\phi_1\rangle} = \sqrt{2 + 2x} + \sqrt{2 - 2x}$$

for $x \in [-1, 1]$. It is an easy calculus problem to find the maximum of this expression. It occurs at $x = 0$, giving $2\sqrt{2}$ as required.