

Mixing doubly stochastic quantum channels with the completely depolarizing channel

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Abstract

It is proved that every doubly stochastic quantum channel that is properly averaged with the completely depolarizing channel can be written as a convex combination of unitary channels. As a consequence, we find that the collection of channels expressible as convex combinations of unitary channels has non-zero Borel measure within the space of doubly stochastic channels.

1 Introduction

Let \mathcal{X} and \mathcal{Y} be finite-dimensional complex Hilbert spaces, and let $L(\mathcal{X})$ and $L(\mathcal{Y})$ denote the sets of linear operators acting on these spaces. By a *super-operator* one means a linear mapping of the form

$$\Phi : L(\mathcal{X}) \rightarrow L(\mathcal{Y}). \quad (1)$$

Such a super-operator is said to be *admissible* if and only if it is both completely positive and preserves trace. In the usual (Schrödinger) picture of quantum information, the admissible super-operators represent valid physical operations, transforming a system having associated space \mathcal{X} to one with associated space \mathcal{Y} . For this reason, admissible super-operators are also commonly referred to as *channels*.

A super-operator Φ is said to be *unital* if it is the case that $\Phi(\mathbb{1}_{\mathcal{X}}) = \mathbb{1}_{\mathcal{Y}}$, for $\mathbb{1}_{\mathcal{X}}$ and $\mathbb{1}_{\mathcal{Y}}$ denoting the identity operators on \mathcal{X} and \mathcal{Y} , respectively. Super-operators that are both unital and admissible are said to be *doubly stochastic*. The existence of a doubly-stochastic channel of the form (1) obviously requires that \mathcal{X} and \mathcal{Y} have equal dimension d , which is assumed hereafter in this paper.

Among the doubly stochastic channels are the *mixed-unitary*¹ channels: those for which there exists a collection of unitary operators $U_1, \dots, U_N \in U(\mathcal{X}, \mathcal{Y})$ and a probability distribution (p_1, \dots, p_N) so that

$$\Phi(X) = \sum_{i=1}^N p_i U_i X U_i^* \quad (2)$$

¹The term *random unitary* channel is commonly used. However, this term is easily confused with a different notion that is now common in quantum information theory—that of a *fixed* unitary operator that is randomly chosen according to some measure (frequently the Haar measure). For this reason, the term *mixed-unitary* channel is suggested in this paper.

for all $X \in L(\mathcal{X})$. (The notation $U(\mathcal{X}, \mathcal{Y})$ refers to the set of unitary mappings from \mathcal{X} to \mathcal{Y} .)

For the case $d = 2$ it holds that every doubly stochastic channel is mixed-unitary, but for larger d this is no longer the case: for all $d \geq 3$ there exist doubly stochastic channels that are not mixed-unitary [Tre86, KM87, LS93]. While it is computationally simple to check that a given channel is doubly stochastic, there is no efficient procedure known to check whether a given channel is mixed-unitary.

Several papers have studied properties of mixed-unitary channels within the last several years. For instance, Gregoratti and Werner [GW03] gave a characterization, with an interesting physical interpretation, of the class of mixed-unitary channels: they proved that a given channel is mixed-unitary if and only if it has full *quantum corrected capacity*, which assumes that the correction procedure is permitted arbitrary measurements on the channel's environment. Smolin, Verstraete, and Winter [SVW05] further investigated this notion of capacity, and conjectured that many copies of any doubly stochastic channel can be closely approximated by a mixed-unitary channel. Buscemi [Bus06] investigated bounds on the number N required in the expression (2) for different channels; Audenaert and Scheel [AS08] investigated conditions under which channels are mixed-unitary; and Mendl and Wolf [MW08] proved several facts about mixed-unitary channels, including the fact that a non-mixed-unitary channel can become mixed-unitary when tensored with the completely depolarizing channel. Rosgen [Ros08] recently proved that various additivity questions about general channels reduce to the same questions on the (potentially simpler) class of mixed-unitary channels.

This paper proves a simple fact about mixed-unitary channels, which is that every doubly stochastic channel becomes mixed-unitary when properly averaged with the completely depolarizing channel. To state this fact more precisely, let us write $\Omega : L(\mathcal{X}) \rightarrow L(\mathcal{Y})$ to denote the completely depolarizing channel, which is defined as

$$\Omega(X) = \frac{\text{Tr}(X)}{d} \mathbb{1}_{\mathcal{Y}}$$

for every operator $X \in L(\mathcal{X})$. This channel is well-known to be mixed-unitary, as it is representable as a uniform mixture over the discrete Weyl operators (also known as the generalized Pauli operators). The claimed fact is now stated in the following theorem.

Theorem 1. *Let \mathcal{X} and \mathcal{Y} be d dimensional complex Hilbert spaces, and let $\Phi : L(\mathcal{X}) \rightarrow L(\mathcal{Y})$ be any doubly stochastic channel. Then for $0 \leq p \leq 1/(d^2 - 1)$ it holds that*

$$p\Phi + (1 - p)\Omega$$

is a mixed-unitary channel.

One corollary of Theorem 1 is that, that within the smallest real affine subspace of super-operators that contains the doubly stochastic channels, there is a ball with positive radius around the completely depolarizing channel within which all super-operators are mixed-unitary channels. This implies that the collection of mixed-unitary channels has non-zero Borel measure within the space of doubly stochastic channels.

Theorem 1 also provides a simple, alternate proof of a fact recently proved by Mendl and Wolf, which is that every doubly stochastic channel can be expressed as a linear combination of unitary channels.

2 Notation and background information

The purpose of this section is to introduce concepts and notation that are needed in the proof of the main result that appears in the following section.

Assume throughout the remainder of the paper that \mathcal{X} and \mathcal{Y} are finite-dimensional complex Hilbert spaces for which a common orthonormal basis $\{|1\rangle, \dots, |d\rangle\}$ has been fixed. We let $L(\mathcal{X}, \mathcal{Y})$ denote the set of all linear operators from \mathcal{X} to \mathcal{Y} , and (as stated above) the notation $L(\mathcal{X})$ is shorthand for $L(\mathcal{X}, \mathcal{X})$ (and likewise for other spaces in place of \mathcal{X}). The usual (Hilbert–Schmidt) inner product of two operators X and Y is defined as

$$\langle X, Y \rangle = \text{Tr}(X^*Y).$$

With respect to the standard basis of \mathcal{X} and \mathcal{Y} , we define a linear bijection

$$\text{vec} : L(\mathcal{X}, \mathcal{Y}) \rightarrow \mathcal{Y} \otimes \mathcal{X}$$

by setting

$$\text{vec}(|i\rangle \langle j|) = |i\rangle \otimes |j\rangle$$

for $1 \leq i, j \leq d$, and extending to all of $L(\mathcal{X}, \mathcal{Y})$ by linearity.

The *Choi–Jamiołkowski representation* [Jam72, Cho75] of a super-operator $\Phi : L(\mathcal{X}) \rightarrow L(\mathcal{Y})$ is defined as

$$J(\Phi) = \sum_{1 \leq i, j \leq d} \Phi(|i\rangle \langle j|) \otimes |i\rangle \langle j|.$$

The mapping J defined in this way is a linear bijection from the space of all super-operators of the form $L(\mathcal{X}) \rightarrow L(\mathcal{Y})$ to the space $L(\mathcal{Y} \otimes \mathcal{X})$. It holds that a super-operator $\Phi : L(\mathcal{X}) \rightarrow L(\mathcal{Y})$ is completely positive if and only if $J(\Phi)$ is positive semidefinite, trace-preserving if and only if $\text{Tr}_{\mathcal{Y}}(J(\Phi)) = \mathbb{1}_{\mathcal{X}}$, and unital if and only if $\text{Tr}_{\mathcal{X}}(J(\Phi)) = \mathbb{1}_{\mathcal{Y}}$. The following facts may be verified directly:

1. For any choice of an operator $A \in L(\mathcal{X}, \mathcal{Y})$, the super-operator $\Phi : L(\mathcal{X}) \rightarrow L(\mathcal{Y})$ defined by $\Phi(X) = AXA^*$ satisfies $J(\Phi) = \text{vec}(A) \text{vec}(A)^*$.
2. For any choice of operators $A \in L(\mathcal{X})$ and $B \in L(\mathcal{Y})$, the super-operator $\Phi : L(\mathcal{X}) \rightarrow L(\mathcal{Y})$ defined by $\Phi(X) = \langle A, X \rangle B$ satisfies $J(\Phi) = B \otimes \overline{A}$.

When combined with the linearity of the mapping J , these facts will allow for simple calculations of $J(\Phi)$ for particular super-operators Φ appearing later in the paper.

Next, let us recall that the *swap operator* on $\mathcal{Y} \otimes \mathcal{X}$ is defined as

$$W = \sum_{1 \leq i, j \leq d} |i\rangle \langle j| \otimes |j\rangle \langle i|,$$

and let us write R and S to denote the orthogonal projections on $\mathcal{Y} \otimes \mathcal{X}$ defined as

$$R = \frac{1}{2}(\mathbb{1} - W) \quad \text{and} \quad S = \frac{1}{2}(\mathbb{1} + W).$$

The projections R and S denote the *anti-symmetric* and *symmetric projections* on $\mathcal{Y} \otimes \mathcal{X}$, respectively. Subscripts are used when it is necessary to be explicit about the spaces on which the projections R and S act; so, as just defined, the operators R and S are written more precisely as $R_{\mathcal{Y} \otimes \mathcal{X}}$ and $S_{\mathcal{Y} \otimes \mathcal{X}}$,

and in general other spaces may be substituted for \mathcal{X} and \mathcal{Y} . In the next section we will make use of the identities

$$\begin{aligned}\mathrm{Tr}_{\mathcal{X}} [(\mathbb{1}_{\mathcal{Y}} \otimes X)R] &= \frac{1}{2} \mathrm{Tr}(X)\mathbb{1}_{\mathcal{Y}} - \frac{1}{2}X, \\ \mathrm{Tr}_{\mathcal{X}} [(\mathbb{1}_{\mathcal{Y}} \otimes X)S] &= \frac{1}{2} \mathrm{Tr}(X)\mathbb{1}_{\mathcal{Y}} + \frac{1}{2}X,\end{aligned}\tag{3}$$

which hold for all $X \in L(\mathcal{X})$.

Finally, it is necessary that a few points on integrals over unitary operators are discussed. Let us first note that the collection of mixed-unitary channels $\Phi : L(\mathcal{X}) \rightarrow L(\mathcal{Y})$ is convex and compact. For any measure ν on the Borel subsets of $U(\mathcal{X}, \mathcal{Y})$, normalized so that $\nu(U(\mathcal{X}, \mathcal{Y})) = 1$, it follows that the super-operator defined as

$$\Phi(X) = \int UXU^* d\nu(U)\tag{4}$$

for every $X \in L(\mathcal{X})$ is a mixed-unitary channel—or in other words can be expressed in the form (2) for some finite choice of N . (It can easily be shown, using Carathéodory's Theorem, that such an expression exists for some choice of $N \leq d^4 - 2d^2 + 2$. See also Buscemi [Bus06] for bounds based on $\mathrm{rank}(J(\Phi))$.)

Hereafter let us write μ to denote the normalized Haar measure on $U(\mathcal{X}, \mathcal{Y})$. This is the unique measure on the Borel subsets of $U(\mathcal{X}, \mathcal{Y})$ that satisfies $\mu(U(\mathcal{X}, \mathcal{Y})) = 1$ and is invariant under left multiplication by every unitary operator $V \in U(\mathcal{Y})$. It is clear that identity

$$\int \mathrm{vec}(U) \mathrm{vec}(U)^* d\mu(U) = \frac{1}{d} \mathbb{1}_{\mathcal{Y} \otimes \mathcal{X}}\tag{5}$$

holds. We will also require the identity

$$\begin{aligned}\int (\mathrm{vec}(U) \mathrm{vec}(U)^* \otimes \mathrm{vec}(U) \mathrm{vec}(U)^*) d\mu(U) \\ = \frac{2}{d(d-1)} R_{\mathcal{Y}_1 \otimes \mathcal{Y}_2} \otimes R_{\mathcal{X}_1 \otimes \mathcal{X}_2} + \frac{2}{d(d+1)} S_{\mathcal{Y}_1 \otimes \mathcal{Y}_2} \otimes S_{\mathcal{X}_1 \otimes \mathcal{X}_2},\end{aligned}\tag{6}$$

where \mathcal{X}_1 and \mathcal{X}_2 represent isomorphic copies of the space \mathcal{X} , \mathcal{Y}_1 and \mathcal{Y}_2 are isomorphic copies of \mathcal{Y} , and we view that

$$\int (\mathrm{vec}(U) \mathrm{vec}(U)^* \otimes \mathrm{vec}(U) \mathrm{vec}(U)^*) d\mu(U) \in L(\mathcal{Y}_1 \otimes \mathcal{X}_1 \otimes \mathcal{Y}_2 \otimes \mathcal{X}_2).$$

The identity (6) may be confirmed by considering the well-known *twirling* operation [Wer89]

$$\int (U \otimes U) X (U \otimes U)^* d\mu(U) = \frac{2}{d(d-1)} \langle R, X \rangle R + \frac{2}{d(d+1)} \langle S, X \rangle S,$$

and taking the Choi-Jamiołkowski representation of both sides. (For the interested reader, it is noted that the above integrals (5) and (6), as well as higher-order variants of them, may also be easily evaluated by means of a general formula of Collins and Śniady [CS06].)

3 The main theorem and two corollaries

We now prove the main theorem, which is restated here for convenience.

Theorem 1. *Let \mathcal{X} and \mathcal{Y} be d dimensional complex Hilbert spaces, and let $\Phi : L(\mathcal{X}) \rightarrow L(\mathcal{Y})$ be any doubly stochastic channel. Then for $0 \leq p \leq 1/(d^2 - 1)$ it holds that*

$$p\Phi + (1 - p)\Omega$$

is a mixed-unitary channel.

Proof. Given that Ω is mixed-unitary, and that the set of mixed-unitary channels is convex, it suffices to consider the case $p = 1/(d^2 - 1)$.

Define a super-operator $\Psi : L(\mathcal{X}) \rightarrow L(\mathcal{Y})$ as

$$\Psi(X) = \int UXU^* \langle \text{vec}(U) \text{vec}(U)^*, J(\Phi) \rangle d\mu(U).$$

By the above identity (5) we have that

$$\int \langle \text{vec}(U) \text{vec}(U)^*, J(\Phi) \rangle d\mu(U) = 1,$$

and it is clear that $\langle \text{vec}(U) \text{vec}(U)^*, J(\Phi) \rangle$ is nonnegative for all $U \in U(\mathcal{X}, \mathcal{Y})$. It follows that Ψ is a mixed-unitary channel. To complete the proof, it will suffice to establish that

$$\Psi = \frac{d^2 - 2}{d^2 - 1}\Omega + \frac{1}{d^2 - 1}\Phi, \quad (7)$$

for then the right-hand-side will be shown to be mixed-unitary as required.

To this end, consider the Choi-Jamiołkowski representation of Ψ , which is

$$J(\Psi) = \int \text{vec}(U) \text{vec}(U)^* \langle \text{vec}(U) \text{vec}(U)^*, J(\Phi) \rangle d\mu(U).$$

By the identity (6) we have

$$\begin{aligned} J(\Psi) &= \frac{2}{d(d-1)} \text{Tr}_{\mathcal{Y}_2 \otimes \mathcal{X}_2} [(R_{\mathcal{Y}_1 \otimes \mathcal{Y}_2} \otimes R_{\mathcal{X}_1 \otimes \mathcal{X}_2}) (\mathbb{1}_{\mathcal{Y}_1 \otimes \mathcal{X}_1} \otimes J(\Phi))] \\ &\quad + \frac{2}{d(d+1)} \text{Tr}_{\mathcal{Y}_2 \otimes \mathcal{X}_2} [(S_{\mathcal{Y}_1 \otimes \mathcal{Y}_2} \otimes S_{\mathcal{X}_1 \otimes \mathcal{X}_2}) (\mathbb{1}_{\mathcal{Y}_1 \otimes \mathcal{X}_1} \otimes J(\Phi))], \end{aligned}$$

where, in this equation, it is viewed that $J(\Phi) \in L(\mathcal{Y}_2 \otimes \mathcal{X}_2)$ and $J(\Psi) \in L(\mathcal{Y}_1 \otimes \mathcal{X}_1)$. (As in the previous section, the spaces $\mathcal{X}_1, \mathcal{X}_2$ and $\mathcal{Y}_1, \mathcal{Y}_2$ are isomorphic copies of \mathcal{X} and \mathcal{Y} , respectively.)

The above expression of $J(\Psi)$ may now be simplified by means of the equations (3). In particular, we have

$$\begin{aligned} J(\Psi) &= \frac{1}{2d(d-1)} [\text{Tr}(J(\Phi))\mathbb{1}_{\mathcal{Y} \otimes \mathcal{X}} - \text{Tr}_{\mathcal{X}}(J(\Phi)) \otimes \mathbb{1}_{\mathcal{X}} - \mathbb{1}_{\mathcal{Y}} \otimes \text{Tr}_{\mathcal{Y}}(J(\Phi)) + J(\Phi)] \\ &\quad + \frac{1}{2d(d+1)} [\text{Tr}(J(\Phi))\mathbb{1}_{\mathcal{Y} \otimes \mathcal{X}} + \text{Tr}_{\mathcal{X}}(J(\Phi)) \otimes \mathbb{1}_{\mathcal{X}} + \mathbb{1}_{\mathcal{Y}} \otimes \text{Tr}_{\mathcal{Y}}(J(\Phi)) + J(\Phi)]. \end{aligned}$$

Making use of the equalities $\text{Tr}_{\mathcal{X}}(J(\Phi)) = \mathbb{1}_{\mathcal{Y}}$ and $\text{Tr}_{\mathcal{Y}}(J(\Phi)) = \mathbb{1}_{\mathcal{X}}$ (which follow from Φ being doubly stochastic), we have

$$J(\Psi) = \frac{d^2 - 2}{d(d^2 - 1)}\mathbb{1}_{\mathcal{Y} \otimes \mathcal{X}} + \frac{1}{d^2 - 1}J(\Phi) = \frac{d^2 - 2}{d^2 - 1}J(\Omega) + \frac{1}{d^2 - 1}J(\Phi).$$

As J is a linear bijection, this implies that equation (7) holds, and therefore completes the proof. \square

As claimed in Section 1, Theorem 1 implies that, within the smallest real affine subspace of super-operators that contains the doubly stochastic channels, there is a ball with positive radius around the completely depolarizing channel within which all super-operators are mixed-unitary. This fact is established by the following corollary.

Corollary 2. *Let \mathcal{X} and \mathcal{Y} be d dimensional complex Hilbert spaces, and suppose $\Phi : L(\mathcal{X}) \rightarrow L(\mathcal{Y})$ is a trace-preserving and unital super-operator such that $J(\Phi)$ is Hermitian and satisfies*

$$\left\| J(\Phi) - \frac{1}{d} \mathbb{1}_{\mathcal{Y} \otimes \mathcal{X}} \right\|_{\infty} \leq \frac{1}{d(d^2 - 1)}.$$

Then Φ is a mixed-unitary channel.

Proof. Define $\Psi = (d^2 - 1)\Phi - (d^2 - 2)\Omega$, which is clearly trace-preserving and unital. It follows from the expression

$$J(\Psi) = (d^2 - 1)(J(\Phi) - J(\Omega)) + J(\Omega) = (d^2 - 1) \left(J(\Phi) - \frac{1}{d} \mathbb{1}_{\mathcal{Y} \otimes \mathcal{X}} \right) + \frac{1}{d} \mathbb{1}_{\mathcal{Y} \otimes \mathcal{X}},$$

together with the assumptions of the corollary, that Ψ is completely positive, and thus is doubly stochastic. By Theorem 1 we therefore have that

$$\frac{d^2 - 2}{d^2 - 1} \Omega + \frac{1}{d^2 - 1} \Psi = \Phi$$

is mixed-unitary, as required. □

Michale Wolf has pointed out to the author that Theorem 1 also provides a simple proof of the following fact, which is part of a characterization of doubly stochastic channels recently proved in [MW08].

Corollary 3 (Mendl and Wolf). *Let \mathcal{X} and \mathcal{Y} be d dimensional complex Hilbert spaces and suppose $\Phi : L(\mathcal{X}) \rightarrow L(\mathcal{Y})$ is a completely positive super-operator. Then Φ is doubly stochastic if and only there exists a collection of unitary operators $U_1, \dots, U_N \in U(\mathcal{X}, \mathcal{Y})$ and a vector of real numbers $(\alpha_1, \dots, \alpha_N)$ with $\sum_{i=1}^N \alpha_i = 1$ such that*

$$\Phi(X) = \sum_{i=1}^N \alpha_i U_i X U_i^* \tag{8}$$

for all $X \in L(\mathcal{X})$.

Proof. It is obvious that any super-operator of the form (8) is unital, and is therefore doubly stochastic assuming it is completely positive. On the other hand, if Φ is doubly stochastic, then by Theorem 1 we have that

$$\Xi = \frac{1}{d^2 - 1} \Phi + \frac{d^2 - 2}{d^2 - 1} \Omega$$

is mixed-unitary. The corollary then follows from the equation

$$\Phi = (d^2 - 1)\Xi - (d^2 - 2)\Omega,$$

along with the fact that both Ξ and Ω are mixed-unitary. □

4 Discussion

As has been discussed by several authors, the Choi-Jamiołkowski representation reveals density operator analogues to various facts about channels. In the present case, the mapping

$$\Phi \mapsto \frac{1}{d}J(\Phi)$$

gives a linear bijection from the set of doubly stochastic channels $\Phi : L(\mathcal{X}) \rightarrow L(\mathcal{Y})$ to the collection of density operators $\rho \in D(\mathcal{Y} \otimes \mathcal{X})$ whose reduced states on both \mathcal{Y} and \mathcal{X} are completely mixed:

$$\mathrm{Tr}_{\mathcal{Y}}(\rho) = \frac{1}{d}\mathbb{1}_{\mathcal{X}} \quad \text{and} \quad \mathrm{Tr}_{\mathcal{X}}(\rho) = \frac{1}{d}\mathbb{1}_{\mathcal{Y}}. \quad (9)$$

Corollary 2 therefore establishes that within the smallest affine subspace that contains all such density operators, there is a ball of positive radius around the maximally mixed state containing only states that are expressible as mixtures of *maximally entangled* states. When expressed in these terms, it is appropriate to compare the results of the previous section with an analogous fact concerning the set of separable states, which are those expressible as mixtures of *unentangled* states.

Specifically, it follows from the results proved in [GB02] that any unit-trace Hermitian operator $\rho \in L(\mathcal{Y} \otimes \mathcal{X})$ satisfying

$$\|\rho - \mathbb{1}/d^2\|_2 \leq \frac{1}{d^2} \quad (10)$$

is a separable density operator. In contrast, Corollary 2 establishes that any unit-trace Hermitian operator $\rho \in L(\mathcal{Y} \otimes \mathcal{X})$ satisfying both the linear constraints (9) and the bound

$$\|\rho - \mathbb{1}/d^2\|_{\infty} \leq \frac{1}{d^2(d^2 - 1)}, \quad (11)$$

is expressible as a convex combination of maximally entangled states. (It is clear that operators not satisfying the constraints (9) are not expressible as convex combinations of maximally entangled states, regardless of their distance from the maximally mixed state.)

It is noted that the ball defined by (11) is properly contained in the one defined by (10). All of the states contained in the smaller ball are therefore representable at two extremes: as convex combinations of unentangled states and convex combinations of maximally entangled states. To what extent this fact can be generalized is an interesting question for future research.

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