Extended Nonlocal Games from Quantum-Classical Games

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Abstract

Several variants of nonlocal games have been considered in the study of quantum entanglement and nonlocality. This paper concerns two of these variants, called *quantum-classical games* and *extended nonlocal games*. We give a construction of an extended nonlocal game from any quantum-classical game that allows one to translate certain facts concerning quantum-classical games to extended nonlocal games. In particular, based on work of Regev and Vidick, we conclude that there exist extended nonlocal games for which no finite-dimensional entangled strategy can be optimal. In the parlance of Einstein-Podolski-Rosen steering, this implies the existence of a tripartite steering inequality for which an infinite-dimensional quantum state is required in order to achieve a maximal violation.

1 Introduction

Various abstract notions of *games* have been considered in the study of entanglement and nonlocality [CHTW04, BBT05, CSUU08, DLTW08, KKM$^+$08, KRT08, Bus12, Fri12, LTW13, TFKW13, CM14, CJPPG15, RV15, JMRW16]. For instance, in a *nonlocal game*, two cooperating players (Alice and Bob) engage in an interaction with a third party (known as the *referee*) [CHTW04]. The referee randomly chooses a pair of questions (x, y) according to a known distribution. Alice receives x, Bob receives y, and without communicating with one another, Alice must respond with an answer a and Bob with an answer b. The referee then evaluates a predicate $P(a, b|x, y)$ to determine whether Alice and Bob win or lose. It is a well-known consequence of earlier work in theoretical physics [Bel64, KS67, CHSH69] that entanglement shared between Alice and Bob can allow them to outperform all purely classical strategies for some nonlocal games. (Nonlocal games were also previously studied in theoretical computer science, in [Raz98] for instance, although generally not by this name and without deference to entanglement or quantum information—but rather as an abstraction of one-round, two-player classical interactive proof systems.)

In a nonlocal game, the referee is classical—it is only the players Alice and Bob that potentially manipulate quantum information. Some generalizations of nonlocal games in which quantum information is exchanged in some way between the players and the referee include ones studied in [Bus12, Fri12, LTW13, TFKW13, CJPPG15, RV15, JMRW16]. In this paper we consider two such generalizations: *quantum-classical games* and *extended nonlocal games*.

1. *Quantum-classical games*. Quantum-classical games, or *QC games* for short, differ from nonlocal games in that the referee begins the game by preparing a tripartite quantum state and sending one part of it to each player, keeping a part for itself. (This step replaces the generation of a classical question pair (x, y) in an ordinary nonlocal game.) The players respond with classical answers a and b as before, and finally the referee determines whether the players win or lose by measuring its part of the original quantum state it initially prepared. (This step replaces the evaluation of a predicate $P(a, b|x, y)$ in an ordinary nonlocal game.)

   Games of this form, with slight variations from the general class just described, were considered by Buscemi [Bus12] and Regev and Vidick [RV15].

2. *Extended nonlocal games*. In an extended nonlocal game, Alice and Bob first present the referee with a quantum system of a fixed size, initialized as Alice and Bob choose, and possibly entangled with systems held by Alice and Bob. (This initialization step generalizes the sharing of entanglement between Alice and Bob in an ordinary nonlocal game, allowing them to give a part of this shared state to the referee.) The game then proceeds much like an ordinary nonlocal game: the referee chooses a pair of (classical) questions (x, y) according to a known distribution, sends x to Alice and y to Bob, and receives a classical answer a from Alice and b from Bob. Finally, to determine whether or not Alice and Bob win, the referee performs a binary-valued measurement, depending on x, y, a, and b, on the system initially sent to it by Alice and Bob. (This measurement replaces the evaluation of the predicate $P(a, b|x, y)$ in an ordinary nonlocal game.)

   Games of this form, again with a slight variation from the general class just described, were considered by Fritz [Fri12], who called them *bipartite steering games*. Extended nonlocal games represent a game-based formulation of the phenomenon of *tripartite steering* investigated in [CSA$^+$15, SBC$^+$15]. (The clash in nomenclature reflects one's view of the referee's role either as a non-player in a game or as a participant in an experiment.) Extended nonlocal games were so-named and studied in [JMRW16], as a means to unify nonlocal games with the *monogamy-of-entanglement games* introduced in [TFKW13].

Regev and Vidick [RV15] proved that certain QC games have the following peculiar property: if Alice and Bob make use of an entangled state of two finite-dimensional quantum systems, initially shared between them, they can never achieve perfect optimality—it is always possible for them to do better (meaning that they win with a strictly larger probability) using some different shared entangled state on two larger quantum systems. Thus, it is only in the limit, as the local dimensions of their shared entangled states goes to infinity, that they can approach an optimal performance in these specific examples of games. This was previously established for analogues of nonlocal games for which both the questions and answers are quantum [LTW13], and it is an open question to determine if the same property holds for any ordinary nonlocal game, where both the questions and answers must be classical. The so-called *I3322 inequality*, when formulated as a nonlocal game, has been conjectured to have the property just described, in which increasing degrees of entanglement admit strategies with strictly increasing success rates [PV10].

In this paper we describe a construction through which any QC game can be transformed into an extended nonlocal game, in such a way that basic properties associated with entangled strategies for the QC game are inherited by the extended nonlocal game. In particular, by applying this construction to the QC games identified by Regev and Vidick, we obtain extended nonlocal games that cannot be played with perfect optimality by Alice and Bob using an entangled state on finite-dimensional systems. In the language of quantum steering, this yields a tripartite steering inequality for which a maximal violation requires infinite-dimensional quantum systems.

It is appropriate to consider this implication in light of Slofstra's recent work [Slo16] resolving one form of *Tsirelson's problem*. With respect to the classes of games discussed above, Slofstra's work concerns the ordinary nonlocal games model—establishing that entangled strategies for Alice and Bob that make use of finite-dimensional systems are strictly weaker than so-called *commuting measurement strategies* on a single, infinite-dimensional quantum system. While the implication of this spectacular result is similar in spirit to the one mentioned above (for the ordinary nonlocal games model no less), Slofstra's work does not imply the existence of a nonlocal game (or an extended nonlocal game) for which increasing amounts of entanglement always yield higher winning probabilities. That is, for any given game there could be a single strategy that is optimal among all finite-dimensional entangled strategies, but where a higher winning probability emerges when one transitions to a commuting measurement strategy.

2 Definitions

We begin with precise definitions of the two classes of games considered in this paper, which are QC games and extended nonlocal games. In addition, we formalize the notions of *entangled strategies* for these games along with their associated *values*, which represent the probabilities that the strategies lead to a win for Alice and Bob.

The reader is assumed to be familiar with standard notions of quantum information, as described in [NC00] and [Wil13], for instance. We will generally follow the terminology and notational conventions of [Wat16]. For example, a *register* X is an abstract quantum system described by a finite-dimensional complex Hilbert space $\mathcal{X}$ having a fixed standard basis $\{|1\rangle, \ldots, |n\rangle\}$ (for some positive integer n); the sets $\mathrm{L}(\mathcal{X})$, $\mathrm{Pos}(\mathcal{X})$, $\mathrm{D}(\mathcal{X})$, and $\mathrm{U}(\mathcal{X})$ denote the set of all linear operators, positive semidefinite operators, density operators, and unitary operators (respectively) acting on such a space $\mathcal{X}$; we write X^*, $\overline{X}$, and X^{T} to refer to the adjoint, entrywise complex conjugate, and transpose of an operator X (with respect to the standard basis in the case of the entrywise complex conjugate and transpose); and $\langle X, Y\rangle = \mathrm{Tr}(X^*Y)$ denotes the Hilbert-Schmidt inner product of operators X and Y.

2.1 Extended nonlocal games

An *extended nonlocal game* is specified by the following objects:

- A probability distribution $\pi : X \times Y \rightarrow [0,1]$, for finite and nonempty sets X and Y.
- A collection of measurement operators $\{P_{a,b,x,y} : a \in A,\ b \in B,\ x \in X,\ y \in Y\} \subset \mathrm{Pos}(\mathcal{R})$, where A and B are finite and nonempty sets and $\mathcal{R}$ is the space corresponding to a register R.

From the referee's perspective, such a game is played as follows:

1. Alice and Bob present the referee with the register R, which has been initialized in a state of Alice and Bob's choosing. (The register R might, for instance, be entangled with systems possessed by Alice and Bob.)
2. The referee randomly generates a pair $(x,y) \in X \times Y$ according to the distribution π, and then sends x to Alice and y to Bob. Alice responds with $a \in A$ and Bob responds with $b \in B$.
3. The referee measures R with respect to the binary-valued measurement $\{P_{a,b,x,y}, \mathbb{1} - P_{a,b,x,y}\}$. The outcome corresponding to the measurement operator $P_{a,b,x,y}$ indicates that Alice and Bob *win*, while the other measurement result indicates that they *lose*.

There are various classes of *strategies* that may be considered for Alice and Bob in an extended nonlocal game, including *unentangled strategies*, *entangled strategies* (or *standard quantum strategies*), and *commuting measurement strategies* [JMRW16]. (Additional classes of strategies, such as *no-signaling strategies*, can also be defined.) In this paper we will only consider *entangled strategies*, in which Alice and Bob begin the game in possession of finite-dimensional quantum systems that have been initialized as they choose. They may then measure these systems in order to obtain answers to the referee's questions.

In more precise terms, an entangled strategy for an extended nonlocal game, specified by $\pi : X \times Y \to [0,1]$ and $\{P_{a,b,x,y} \;:\; a \in A,\, b \in B,\, x \in X,\, y \in Y\} \subset \mathrm{Pos}(\mathcal{R})$ as above, consists of these objects:

1. A state $\sigma \in \mathrm{D}(\mathcal{U} \otimes \mathcal{R} \otimes \mathcal{V})$, for $\mathcal{U}$ being the space corresponding to a register U held by Alice and $\mathcal{V}$ being the space corresponding to a register V held by Bob. This state represents Alice and Bob's initialization of the triple $(\mathsf{U}, \mathsf{R}, \mathsf{V})$ immediately before R is sent to the referee.
2. A measurement $\{A_a^x \;:\; a \in A\} \subset \mathrm{Pos}(\mathcal{U})$ for each $x \in X$, performed by Alice when she receives the question x, and a measurement $\{B_b^y \;:\; b \in B\} \subset \mathrm{Pos}(\mathcal{V})$ for each $y \in Y$, performed by Bob when he receives the question y.

When Alice and Bob utilize such a strategy, their winning probability p may be expressed as

$$p = \sum_{\substack{(x,y) \in X \times Y \\ (a,b) \in A \times B}} \pi(x,y) \big\langle A_a^x \otimes P_{a,b,x,y} \otimes B_b^y, \sigma \big\rangle. \tag{1}$$

The *entangled value* of an extended nonlocal game represents the supremum of the winning probabilities, taken over all entangled strategies. If H is the name assigned to an extended nonlocal game having a specification as above, then we write $\omega_N^*(H)$ to denote the *maximum* winning probability taken over all entangled strategies for which $\dim(\mathcal{U} \otimes \mathcal{V}) \leq N$, so that the entangled value of H is

$$\omega^*(H) = \lim_{N \to \infty} \omega_N^*(H). \tag{2}$$

2.2 Quantum-classical games

A *quantum-classical game* (or *QC game*, for short) is specified by the following objects:

- A state $\rho \in \mathrm{D}(\mathcal{X} \otimes \mathcal{S} \otimes \mathcal{Y})$ of a triple of registers $(\mathsf{X}, \mathsf{S}, \mathsf{Y})$.
- A collection of measurement operators $\{Q_{a,b} \;:\; a \in A,\; b \in B\} \subset \mathrm{Pos}(\mathcal{S})$, for finite and nonempty sets A and B.

From the referee's perspective, such a game is played as follows:

1. The referee prepares $(\mathsf{X}, \mathsf{S}, \mathsf{Y})$ in the state ρ, then sends X to Alice and Y to Bob.
2. Alice responds with $a \in A$ and Bob responds with $b \in B$.
3. The referee measures S with respect to the binary-valued measurement $\{Q_{a,b}, \mathbb{1} - Q_{a,b}\}$. The outcome corresponding to the measurement operator $Q_{a,b}$ indicates that Alice and Bob *win*, while the other measurement result indicates that they *lose*.

Similar to extended nonlocal games, one may consider various classes of strategies for QC games. Again, we will consider only entangled strategies, in which Alice and Bob begin the game in possession of finite-dimensional quantum systems initialized as they choose.

More precisely, an entangled strategy for a QC game, specified by $\rho \in \mathrm{D}(\mathcal{X} \otimes \mathcal{S} \otimes \mathcal{Y})$ and $\{Q_{a,b} \;:\; a \in A,\; b \in B\} \subset \mathrm{Pos}(\mathcal{S})$ as above, consists of these objects:

1. A state $\sigma \in \mathrm{D}(\mathcal{U} \otimes \mathcal{V})$, for $\mathcal{U}$ being the space corresponding to a register U held by Alice and $\mathcal{V}$ being the space corresponding to a register V held by Bob.
2. A measurement $\{A_a : a \in A\} \subset \mathrm{Pos}(\mathcal{U} \otimes \mathcal{X})$ for Alice, performed on the pair (U, X) after she receives X from the referee, and a measurement $\{B_b : b \in B\} \subset \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{V})$ for Bob, performed on the pair (Y, V) after he receives Y from the referee.

The winning probability of such a strategy may be expressed as

$$p = \sum_{(a,b) \in A \times B} \langle A_a \otimes Q_{a,b} \otimes B_b, W(\sigma \otimes \rho) W^* \rangle, \tag{3}$$

where W is the unitary operator that corresponds to the natural re-ordering of registers consistent with each of the tensor product operators $A_a \otimes Q_{a,b} \otimes B_b$ (i.e., the permutation $(\mathsf{U}, \mathsf{V}, \mathsf{X}, \mathsf{S}, \mathsf{Y}) \mapsto (\mathsf{U}, \mathsf{X}, \mathsf{S}, \mathsf{Y}, \mathsf{V})$).

3 Construction and analysis

In this section we will describe a construction of an extended nonlocal game from any given QC game, and analyze the relationship between the constructed extended nonlocal game and the original QC game.

3.1 Construction

Suppose that a QC game G, specified by a state $\rho \in \mathrm{D}(\mathcal{X} \otimes \mathcal{S} \otimes \mathcal{Y})$ and a collection of measurement operators $\{Q_{a,b} : a \in A, \ b \in B\} \subset \mathrm{Pos}(\mathcal{S})$, is given. We construct an extended nonlocal game H as follows:

1. Let $n = \dim(\mathcal{X})$ and $m = \dim(\mathcal{Y})$, let

$$X = \{1, \ldots, n^2\} \quad \text{and} \quad Y = \{1, \ldots, m^2\}, \tag{4}$$

   and let $\pi : X \times Y \to [0,1]$ be the uniform probability distribution on these sets, so that $\pi(x,y) = n^{-2} m^{-2}$ for every $x \in X$ and $y \in Y$.

2. Let $\mathsf{R} = (\mathsf{X}, \mathsf{Y})$, define

$$\xi = \mathrm{Tr}_{\mathcal{S}}(\rho) \quad \text{and} \quad \xi_{a,b} = \mathrm{Tr}_{\mathcal{S}}\big[(\mathbb{1}_{\mathcal{X}} \otimes Q_{a,b} \otimes \mathbb{1}_{\mathcal{Y}})\rho\big] \tag{5}$$

   for each $a \in A$ and $b \in B$, let

$$\{U_1, \ldots, U_{n^2}\} \subset \mathrm{U}(\mathcal{X}) \quad \text{and} \quad \{V_1, \ldots, V_{m^2}\} \subset \mathrm{U}(\mathcal{Y}) \tag{6}$$

   be any fixed choice of orthogonal sets of unitary operators (e.g., the discrete Weyl operators, described in [DFH06] for instance), and let

$$P_{a,b,x,y} = \mathbb{1}_{\mathcal{X}} \otimes \mathbb{1}_{\mathcal{Y}} - (U_x \otimes V_y)(\xi^{\mathsf{T}} - \xi_{a,b}^{\mathsf{T}})(U_x \otimes V_y)^* \tag{7}$$

   for every $a \in A$, $b \in B$, $x \in X$, and $y \in Y$.

One may observe that $P_{a,b,x,y}$ is indeed a measurement operator for each $a \in A$, $b \in B$, $x \in X$, and $y \in Y$, meaning that $0 \leq P_{a,b,x,y} \leq \mathbb{1}_{\mathcal{X}} \otimes \mathbb{1}_{\mathcal{Y}}$, by virtue of the fact that $0 \leq \xi_{a,b} \leq \xi \leq \mathbb{1}$ for every $a \in A$ and $b \in B$.

The basic intuition behind this construction is as follows. In the game G, the referee sends X to Alice and Y to Bob, but in the game H it is Alice and Bob that give X and Y to the referee. To simulate, within the game H, the sort of transmission that occurs in G, it is natural to consider *teleportation*—for if Alice provided the referee with the register X in a state maximally entangled with a register of her own, and Bob did likewise with Y, then the referee could effectively teleport a copy of X to Alice and a copy of Y to Bob. Now, in an extended nonlocal game, the referee cannot actually perform teleportation in this way: the question pair (x, y) needs to be randomly generated, independent of the state of the registers (X, Y). For this reason the game H is based on a form of *post-selected teleportation*, where x and y are chosen randomly, and then later compared with hypothetical measurement results that would be obtained if the referee were to perform teleportation. The details of the construction above result from a combination of this idea together with algebraic simplifications.

3.2 Game values

It is not immediate that the construction above should necessarily translate the basic properties of the game G to the game H; Alice and Bob are free to behave as they choose, which is not necessarily consistent with the intuitive description of the game H based on teleportation suggested above. An analysis does, however, reveal that the construction works as one would hope (and perhaps expect). In particular, we will prove two bounds on the value of the extended nonlocal game H constructed from a QC game G as described above:

$$\omega^*_{nmN}(H) \geq 1 - \frac{1 - \omega^*_N(G)}{nm} \quad \text{and} \quad \omega^*_N(H) \leq 1 - \frac{1 - \omega^*_{nmN}(G)}{nm}, \tag{8}$$

for every positive integer N. This implies that

$$\omega^*(H) = 1 - \frac{1 - \omega^*(G)}{nm}. \tag{9}$$

Moreover, H inherits the same limiting behavior of G with respect to entangled strategies, meaning that if $\omega^*_N(G) < \omega^*(G)$ for all $N \in \mathbb{N}$, then $\omega^*_N(H) < \omega^*(H)$ for all $N \in \mathbb{N}$ as well.

We will begin with the first inequality in (8). Assume that an arbitrary strategy for Alice and Bob in the game G is fixed: Alice and Bob make use of a shared entangled state $\sigma \in \mathrm{D}(\mathcal{U} \otimes \mathcal{V})$, where $\dim(\mathcal{U} \otimes \mathcal{V}) \leq N$, and their measurements are given by

$$\{A_a : a \in A\} \subset \mathrm{Pos}(\mathcal{U} \otimes \mathcal{X}) \quad \text{and} \quad \{B_b : b \in B\} \subset \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{V}), \tag{10}$$

respectively. The winning probability of this strategy in the game G may be expressed as

$$p = \sum_{(a,b) \in A \times B} \big\langle A_a \otimes Q_{a,b} \otimes B_b, W(\sigma \otimes \rho) W^* \big\rangle, \tag{11}$$

as was mentioned above, while the losing probability equals

$$q = \sum_{(a,b) \in A \times B} \big\langle A_a \otimes (\mathbb{1} - Q_{a,b}) \otimes B_b, W(\sigma \otimes \rho) W^* \big\rangle = 1 - p. \tag{12}$$

We adapt this strategy to obtain one for H as follows:

1. Alice will hold a register X', representing a copy of X, and Bob will hold Y', representing a copy of Y. The initial state of the register pairs (X',X) and (Y',Y) are to be the canonical maximally entangled states

$$|\psi\rangle = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}|j\rangle|j\rangle \quad \text{and} \quad |\phi\rangle = \frac{1}{\sqrt{m}}\sum_{k=1}^{m}|k\rangle|k\rangle, \tag{13}$$

respectively, where n and m are the dimensions of the spaces corresponding to the registers X and Y. In addition, Alice holds the register U and Bob holds the register V, with (U,V) being prepared in the same shared entangled state σ that is used in the strategy for G.

2. Upon receiving the question $x \in X$ from the referee, Alice performs the unitary operation $\overline{U_x}$ on X', then measures (U,X') with respect to the measurement $\{A_a : a \in A\}$ to obtain an answer $a \in A$. Similarly, upon receiving $y \in Y$ from the referee, Bob performs $\overline{V_y}$ on Y', then measures (Y',V) with respect to $\{B_b : b \in B\}$ to obtain an answer $b \in B$.

The performance of this strategy can be analyzed by first ignoring the specific initialization of the registers described in step 1, and defining a measurement $\{R_0, R_1\}$ that determines, for an arbitrary initialization of these registers, whether Alice and Bob win or lose by behaving as described in step 2. In particular, the measurement $\{R_0, R_1\}$ is defined on the register tuple $(\mathsf{U},\mathsf{X}',\mathsf{X},\mathsf{Y},\mathsf{Y}',\mathsf{V})$, the measurement operator R_0 corresponds to a losing outcome, and R_1 corresponding to a winning outcome. These operators may be described as follows:

$$\begin{aligned}
R_0 &= \frac{1}{n^2m^2}\sum_{\substack{(x,y)\in X\times Y\\(a,b)\in A\times B}}(\mathbb{1}_{\mathcal{U}}\otimes U_x^{\mathsf{T}})A_a(\mathbb{1}_{\mathcal{U}}\otimes\overline{U_x})\otimes(\mathbb{1}_{\mathcal{X}\otimes\mathcal{Y}}-P_{a,b,x,y})\otimes(V_y^{\mathsf{T}}\otimes\mathbb{1}_{\mathcal{V}})B_b(\overline{V_y}\otimes\mathbb{1}_{\mathcal{V}})\\
R_1 &= \frac{1}{n^2m^2}\sum_{\substack{(x,y)\in X\times Y\\(a,b)\in A\times B}}(\mathbb{1}_{\mathcal{U}}\otimes U_x^{\mathsf{T}})A_a(\mathbb{1}_{\mathcal{U}}\otimes\overline{U_x})\otimes P_{a,b,x,y}\otimes(V_y^{\mathsf{T}}\otimes\mathbb{1}_{\mathcal{V}})B_b(\overline{V_y}\otimes\mathbb{1}_{\mathcal{V}}) = \mathbb{1}-R_0.
\end{aligned} \tag{14}$$

Now we may consider the initialization of the registers described in step 1. For an arbitrary choice of operators $X \in \mathrm{L}(\mathcal{U})$ and $Y \in \mathrm{L}(\mathcal{V})$ we have

$$\big\langle R_0, X\otimes|\psi\rangle\langle\psi|\otimes|\phi\rangle\langle\phi|\otimes Y\big\rangle = \sum_{(a,b)\in A\times B}\big\langle A_a\otimes(\xi^{\mathsf{T}}-\xi_{a,b}^{\mathsf{T}})\otimes B_b, X\otimes|\psi\rangle\langle\psi|\otimes|\phi\rangle\langle\phi|\otimes Y\big\rangle, \tag{15}$$

by virtue of the fact that $\big(\overline{U_x}\otimes U_x\big)|\psi\rangle = |\psi\rangle$ and $\big(\overline{V_y}\otimes V_y\big)|\phi\rangle = |\phi\rangle$ for every $x \in X$ and $y \in Y$. Further simplifying this expression, one obtains

$$\begin{aligned}
&\sum_{(a,b)\in A\times B}\big\langle A_a\otimes(\xi^{\mathsf{T}}-\xi_{a,b}^{\mathsf{T}})\otimes B_b, X\otimes|\psi\rangle\langle\psi|\otimes|\phi\rangle\langle\phi|\otimes Y\big\rangle\\
&\qquad = \frac{1}{nm}\sum_{(a,b)\in A\times B}\big\langle A_a\otimes B_b, X\otimes(\xi-\xi_{a,b})\otimes Y\big\rangle\\
&\qquad = \frac{1}{nm}\sum_{(a,b)\in A\times B}\big\langle A_a\otimes(\mathbb{1}-Q_{a,b})\otimes B_b, X\otimes\rho\otimes Y\big\rangle.
\end{aligned} \tag{16}$$

By expressing the initial state σ of (U,V) as $\sigma = \sum_i X_i\otimes Y_i$ and making use of the bilinearity of the above expression in X and Y, one finds that the losing probability of Alice and Bob's strategy for H is equal to $q/(nm)$, for q being the losing probability (12) for their original strategy for G.

Optimizing over all strategies for G that make use of an initial shared state having total dimension at most N yields the required inequality

$$\omega^*_{nmN}(H) \geq 1 - \frac{1 - \omega^*_N(G)}{nm}. \tag{17}$$

Next we will prove the second inequality in (8). Assume that an arbitrary strategy for Alice and Bob in the extended nonlocal game H constructed from G is fixed: the strategy consists of an initial state $\sigma \in \mathrm{D}(\mathcal{U} \otimes (\mathcal{X} \otimes \mathcal{Y}) \otimes \mathcal{V})$ for the registers $(\mathsf{U}, (\mathsf{X}, \mathsf{Y}), \mathsf{V})$, where $\dim(\mathcal{U} \otimes \mathcal{V}) \leq N$, along with measurements

$$\left\{ A^x_a \,:\, a \in A \right\} \subset \mathrm{Pos}(\mathcal{U}) \quad \text{and} \quad \left\{ B^y_b \,:\, b \in B \right\} \subset \mathrm{Pos}(\mathcal{V}) \tag{18}$$

for Alice and Bob, respectively, for each $x \in X$ and $y \in Y$. The winning probability of this strategy may be expressed as

$$p = \frac{1}{n^2 m^2} \sum_{\substack{(x,y) \in X \times Y \\ (a,b) \in A \times B}} \left\langle A^x_a \otimes P_{a,b,x,y} \otimes B^y_b, \sigma \right\rangle \tag{19}$$

while the losing probability is

$$q = \frac{1}{n^2 m^2} \sum_{\substack{(x,y) \in X \times Y \\ (a,b) \in A \times B}} \left\langle A^x_a \otimes (\mathbb{1} - P_{a,b,x,y}) \otimes B^y_b, \sigma \right\rangle = 1 - p. \tag{20}$$

We adapt this strategy to give one for G as follows:

1. Let X' and Y' represent copies of the registers X and Y. Alice and Bob will initially share the registers $(\mathsf{U}, \mathsf{X}', \mathsf{Y}', \mathsf{V})$ initialized to the state $\overline{\sigma}$, with Alice holding $(\mathsf{U}, \mathsf{X}')$ and Bob holding $(\mathsf{Y}', \mathsf{V})$.
2. Upon receiving X from the referee, Alice first measures the pair $(\mathsf{X}', \mathsf{X})$ with respect to the basis $\left\{ (\mathbb{1} \otimes U^*_x) |\psi\rangle \,:\, x \in X \right\}$. For whichever outcome $x \in X$ she obtains, she then measures U with respect to the measurement

$$\left\{ \overline{A^x_a} \,:\, a \in A \right\} \subset \mathrm{Pos}(\mathcal{U}) \tag{21}$$

to obtain an outcome $a \in A$. Bob does likewise, first measuring $(\mathsf{Y}', \mathsf{Y})$ with respect to the basis $\left\{ (\mathbb{1} \otimes V^*_y) |\phi\rangle \,:\, y \in Y \right\}$, and then measuring V with respect to the measurement

$$\left\{ \overline{B^y_b} \,:\, b \in B \right\} \subset \mathrm{Pos}(\mathcal{V}) \tag{22}$$

for whichever outcome $y \in Y$ is obtained.

Now let us consider the probability with which this strategy wins in G. The state of the registers $(\mathsf{U}, \mathsf{X}', \mathsf{X}, \mathsf{S}, \mathsf{Y}, \mathsf{Y}', \mathsf{V})$ immediately after the referee sends X to Alice and Y to Bob is given by

$$W(\overline{\sigma} \otimes \rho) W^*, \tag{23}$$

where W is a unitary operator that corresponds to a permutation of registers:

$$(\mathsf{U}, \mathsf{X}', \mathsf{Y}', \mathsf{V}, \mathsf{X}, \mathsf{S}, \mathsf{Y}) \mapsto (\mathsf{U}, \mathsf{X}', \mathsf{X}, \mathsf{S}, \mathsf{Y}, \mathsf{Y}', \mathsf{V}). \tag{24}$$

We may define a measurement $\{R_0, R_1\}$ on the register tuple $(\mathsf{U}, \mathsf{X}', \mathsf{X}, \mathsf{S}, \mathsf{Y}, \mathsf{Y}', \mathsf{V})$ representing the outcome of the game, with R_0 corresponding to a losing outcome and R_1 corresponding to a winning outcome. We have

$$\begin{aligned} R_0 &= \sum_{\substack{(x,y)\in X\times Y\\(a,b)\in A\times B}} \overline{A^x_a} \otimes (\mathbb{1}\otimes U_x^*)|\psi\rangle\langle\psi|(\mathbb{1}\otimes U_x) \otimes (\mathbb{1} - Q_{a,b}) \otimes (V_y^*\otimes\mathbb{1})|\phi\rangle\langle\phi|(V_y\otimes\mathbb{1}) \otimes \overline{B^y_b} \\ R_1 &= \sum_{\substack{(x,y)\in X\times Y\\(a,b)\in A\times B}} \overline{A^x_a} \otimes (\mathbb{1}\otimes U_x^*)|\psi\rangle\langle\psi|(\mathbb{1}\otimes U_x) \otimes Q_{a,b} \otimes (V_y^*\otimes\mathbb{1})|\phi\rangle\langle\phi|(V_y\otimes\mathbb{1}) \otimes \overline{B^y_b}. \end{aligned} \tag{25}$$

Simplifying expressions for the probability that Alice and Bob lose yields

$$\left\langle R_0, W(\overline{\sigma}\otimes\rho)W^*\right\rangle = \frac{1}{nm}\sum_{\substack{(x,y)\in X\times Y\\(a,b)\in A\times B}} \left\langle A^x_a \otimes (\mathbb{1} - P_{a,b,x,y}) \otimes B^y_b, \sigma\right\rangle = nmq, \tag{26}$$

for q being the losing probability (20) for their original strategy for H.

Optimizing over all strategies for H that make use of an initial shared state for which Alice and Bob's total dimension is at most N yields the inequality

$$\omega^*_N(H) \leq 1 - \frac{1-\omega^*_{nmN}(G)}{nm}. \tag{27}$$

4 Discussion

As was mentioned in the introduction, Regev and Vidick [RV15] have identified examples of QC games for which Alice and Bob can never achieve optimality by using a finite-dimensional entangled strategy. To be more precise, they prove that there exists a QC game[1] G (and in fact a family of such games) for which it holds that $\omega^*_N(G) < 1$ for all $N \in \mathbb{N}$, while $\omega^*(G) = 1$. By applying our construction to any such game, we obtain an extended nonlocal game H with the property that $\omega^*_N(H) < 1$ for all $N \in \mathbb{N}$, while $\omega^*(H) = 1$.

In greater detail, by taking the simplest example of a QC game G with the property just described, and applying our construction (along with minor simplifications), one obtains an extended nonlocal game as follows:

1. Let $\mathcal{X} = \mathcal{Y} = \mathbb{C}^3$ and let $U_1, \ldots, U_9$ be the discrete Weyl operators acting on $\mathbb{C}^3$. Also define

$$\begin{aligned} |\gamma_0\rangle &= \frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{2}|1\rangle|1\rangle + \frac{1}{2}|2\rangle|2\rangle, \\ |\gamma_1\rangle &= \frac{1}{\sqrt{2}}|0\rangle|0\rangle - \frac{1}{2}|1\rangle|1\rangle - \frac{1}{2}|2\rangle|2\rangle. \end{aligned} \tag{28}$$

2. Alice and Bob give a pair of registers (X, Y) to the referee, initialized as they choose. The referee randomly chooses $x, y \in \{1, \ldots, 9\}$ uniformly and independently at random, then sends x to Alice and y to Bob. Alice and Bob respond with binary values $a, b \in \{0, 1\}$, respectively.

[1] Their games fall into a category of QC games that they call *quantum XOR games*, in which $A = B = \{0, 1\}$ and only the parity $a \oplus b$ of Alice and Bob's answers is relevant to the referee's determination of whether they win or lose.

3. The referee computes $c = a \oplus b$, then measures the pair (X, Y) with respect to the measurement

$$\left\{\mathbb{1}_{\mathcal{X}} \otimes \mathbb{1}_{\mathcal{Y}} - (U_x \otimes U_y)|\gamma_c\rangle\langle\gamma_c|(U_x \otimes U_y)^*,\ (U_x \otimes U_y)|\gamma_c\rangle\langle\gamma_c|(U_x \otimes U_y)^*\right\}. \tag{29}$$

The first outcome represents a win for Alice and Bob, and the second a loss. (Note that here we have scaled the losing measurement operator by a factor of two in comparison to what is described in the construction, which has the effect of doubling the losing probability for every strategy of Alice and Bob.)

Assuming Alice and Bob initially entangle the pair (X, Y) with finite-dimensional registers of their own, they can never win the game with certainty, but they can approach certainty by using increasingly large systems.

A natural question that arises from this work is whether a smaller extended nonlocal game has a similar property. Proving an analogous result for ordinary nonlocal games rather than extended nonlocal games remains a central open question in this area.

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